Percentile Queries in Multi-Dimensional Markov Decision Processes

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The talk in one slide

Strategy synthesis for Markov Decision Processes (MDPs)

Finding **good** controllers for systems interacting with a *stochastic* environment.

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- Good? Performance evaluated through *payoff functions*.
- Usual problem is to optimize the *expected performance* or the *probability of achieving a given performance level*.
- Not sufficient for many practical applications.
  - ▷ Reason about *trade-offs* and *interplays*.
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  - ▷ Several extensions, more expressive but also more complex...

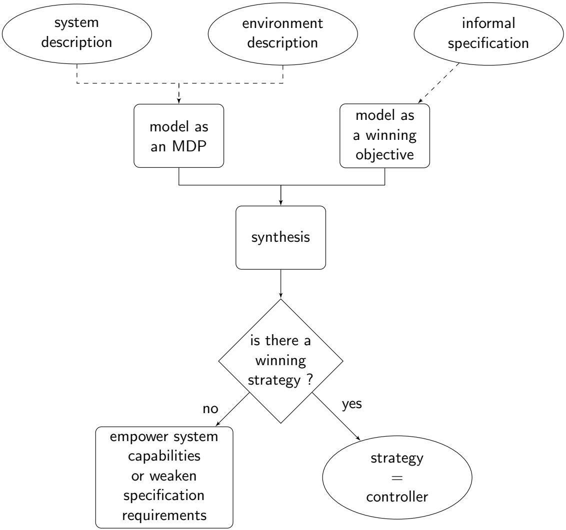
Aim of this talk

Multi-constraint percentile queries: generalizes the problem to multiple dimensions, multiple constraints.

1 Context, MDPs, Strategies

2 Percentile Queries

Strategy (policy) synthesis for MDPs



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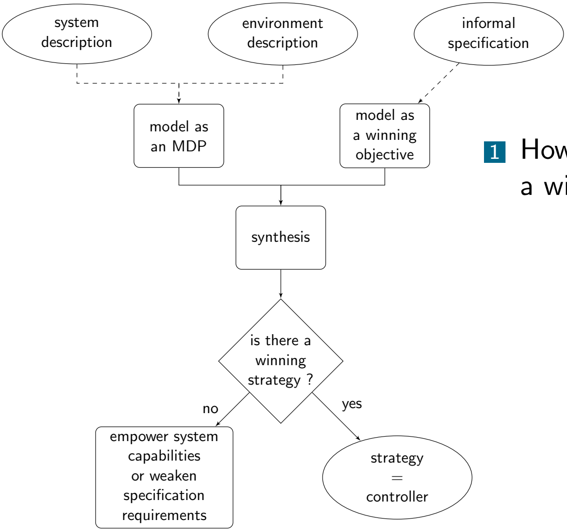

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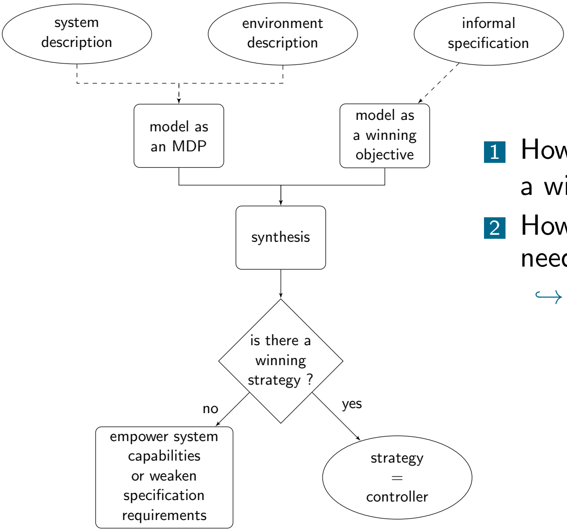

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2. How complex such a strategy needs to be?
   - ↪ **Simpler is better.**

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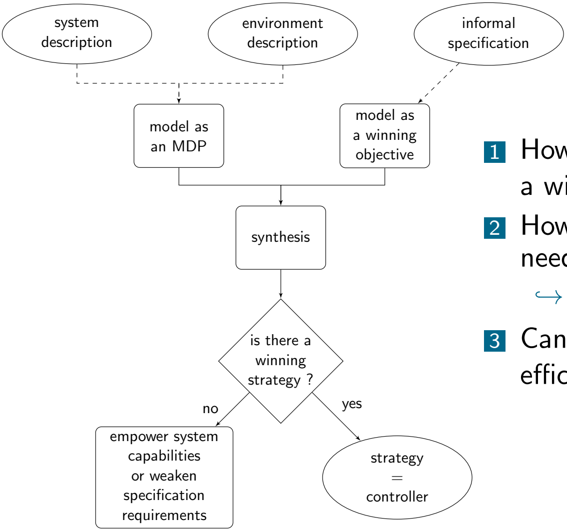
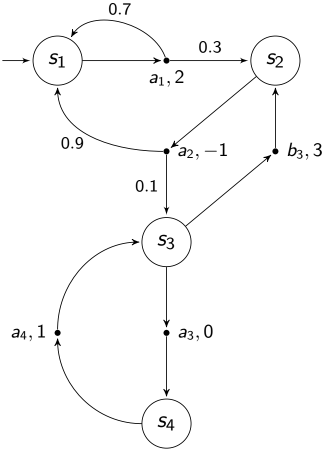


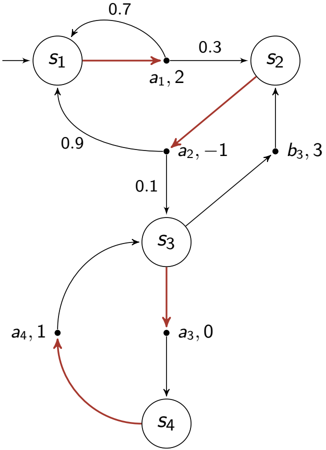
1. How complex is it to decide if a winning strategy exists?
2. How complex such a strategy needs to be?
   ↪ **Simpler is better.**
3. Can we synthesize one efficiently?

Markov decision processes



- **MDP** $M = (S, A, \delta, w)$
  - ▷ finite sets of states S and actions A
  - ▷ probabilistic transition $\delta \colon S \times A \to \mathcal{D}(S)$
  - ▷ weight function $w \colon A \to \mathbb{Z}^d$
- **Run** (or play): $\rho = s_1 a_1 \ldots a_{n-1} s_n \ldots$ such that $\delta(s_i, a_i, s_{i+1}) > 0$ for all $i \geq 1$
  - ▷ set of runs $\mathcal{R}(M)$
  - ▷ set of histories (finite runs) $\mathcal{H}(M)$
- **Strategy** $\sigma \colon \mathcal{H}(M) \to \mathcal{D}(A)$
  - ▷ $\forall\, h$ ending in s, $\mathrm{Supp}(\sigma(h)) \in A(s)$

Markov decision processes



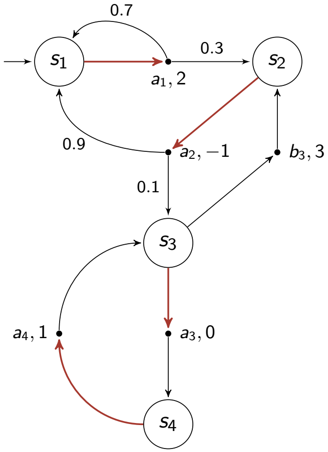
Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2 a_2 s_1 a_1 s_2 a_2 (s_3 a_3 s_4 a_4)^\omega$

Other possible run $\rho' = s_1 a_1 s_2 a_2 (s_3 a_3 s_4 a_4)^\omega$

- Strategies may use
  - finite or infinite **memory**
  - **randomness**
- **Payoff functions** map runs to numerical values
  - truncated sum up to $T = \{s_3\}$: $\mathsf{TS}^T(\rho) = 2$, $\mathsf{TS}^T(\rho') = 1$
  - mean-payoff: $\underline{\mathsf{MP}}(\rho) = \underline{\mathsf{MP}}(\rho') = 1/2$
  - many more

Markov chains



Once initial state s_{init} and strategy σ fixed, fully stochastic process

$\rightsquigarrow$ **Markov chain** (**MC**)

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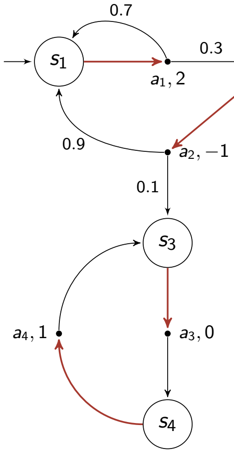

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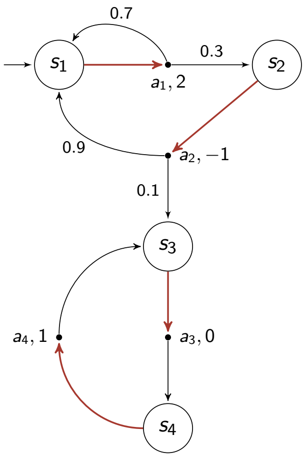

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- Event $\mathcal{E} \subseteq \mathcal{R}(M)$
  - ▷ probability $\mathbb{P}^{\sigma}_{M,s_{\text{init}}}(\mathcal{E})$
- Measurable $f\colon \mathcal{R}(M) \to (\mathbb{R} \cup \{-\infty, \infty\})^d$
  - ▷ expected value $\mathbb{E}^{\sigma}_{M,s_{\text{init}}}(f)$

Single-constraint percentile problem

Ensuring a given performance level with sufficient probability

- uni-dimensional weight function $w\colon A \to \mathbb{Z}$ and payoff function $f\colon \mathcal{R}(M) \to \mathbb{R} \cup \{-\infty, \infty\}$
- well-studied for various payoffs

Single-constraint percentile problem

Given MDP $M = (S, A, \delta, w)$, initial state s_{init}, payoff function f, value threshold $v \in \mathbb{Q}$, and probability threshold $\alpha \in [0,1] \cap \mathbb{Q}$, decide if there exists a strategy σ such that

$$\mathbb{P}^{\sigma}_{M,s_{\mathsf{init}}}\left[\{\rho \in \mathcal{R}_{s_{\mathsf{init}}}(M) \mid f(\rho) \geq v\}\right] \geq \alpha.$$

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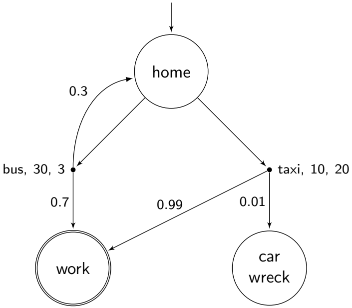
- **percentile constraint**, shortened $\mathbb{P}^{\sigma}_{M,s_{\text{init}}}\left[f \geq v\right] \geq \alpha$

Illustration: stochastic shortest path problem



Two-dimensional weights on actions: *time* and *cost*.

Often necessary to consider trade-offs: e.g., between the probability to reach work in due time and the risks of an expensive journey.

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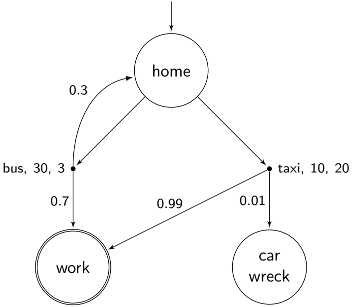

Classical problem considers only a **single percentile constraint**.

- **C1**: 80% of runs reach work in at most 40 minutes.
  - ▷ Taxi $\rightsquigarrow \leq 10$ minutes with probability $0.99 > 0.8$.

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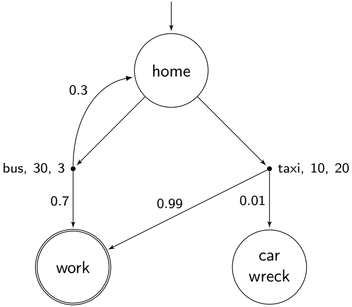

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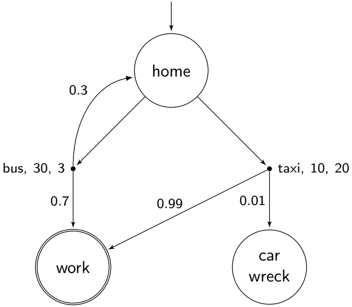

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Taxi $\not\models$ C2, bus $\not\models$ C1. What if we want C1 $\wedge$ C2?

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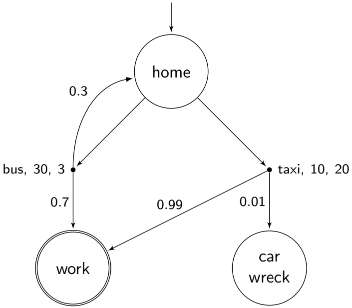

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Study of **multi-constraint percentile queries**.

- ▷ Sample strategy: bus once, then taxi. Requires *memory*.
- ▷ Another strategy: bus with probability 3/5, taxi with probability 2/5. Requires *randomness*.

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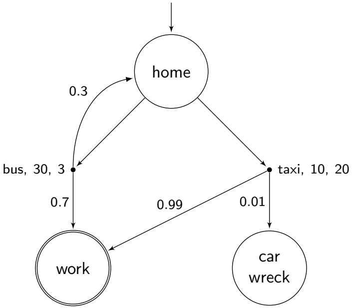

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Study of **multi-constraint percentile queries**.

In general, *both* memory *and* randomness are required.

$\neq$ classical problems (single constraint, expected value, etc)

Multi-constraint percentile problem

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Given d-dimensional MDP $M = (S, A, \delta, w)$, initial state s_{init}, payoff function f, and $q \in \mathbb{N}$ percentile constraints described by dimensions $l_i \in \{1, \ldots, d\}$, value thresholds $v_i \in \mathbb{Q}$ and probability thresholds $\alpha_i \in [0, 1] \cap \mathbb{Q}$, where $i \in \{1, \ldots, q\}$, decide if there exists a strategy σ such that query $\mathcal{Q}$ holds, with

$$\mathcal{Q} := \bigwedge_{i=1}^{q} \mathbb{P}^{\sigma}_{M,s_{\text{init}}}\left[f_{l_i} \geq v_i\right] \geq \alpha_i.$$

Very general framework allowing for: multiple constraints related to $\neq$ or $=$ dimensions, $\neq$ value and probability thresholds.

- $\rightsquigarrow$ For SP, even $\neq$ targets for each constraint.
- $\rightsquigarrow$ Great flexibility in modeling applications.

Results overview (1/2)

- **Wide range of payoff functions**
  - multiple reachability,
  - mean-payoff ($\overline{\mathsf{MP}}$, $\underline{\mathsf{MP}}$),
  - discounted sum (DS).
  - $\inf$, $\sup$, $\liminf$, $\limsup$,
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- **Several variants**:
  - multi-dim. multi-constraint,
  - single-dim. multi-constraint,
  - single-constraint.
- For each one:
  - algorithms,
  - lower bounds,
  - memory requirements.

$\leadsto$ **Complete picture** for this new framework.

Results overview (2/2)

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	$P(M)\cdot E(\mathcal{Q})$ [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	$P(M)\cdot E(\mathcal{Q})$ PSPACE-h.
$\overline{MP}$	P [Put94]	P	P
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- $\mathcal{F} = \{\inf, \sup, \liminf, \limsup\}$
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In most cases, only **polynomial in the model size**.

▷ In practice, the query size can often be bounded while the model can be very large.

Some related work

- **Same philosophy** (i.e., beyond uni-dimensional $\mathbb{E}$ or $\mathbb{P}$ maximization), $\neq$ approaches.
  - Beyond worst-case synthesis: $\mathbb{E}$ + worst-case [BFRR14b].
  - Survey of recent extensions in VMCAI'15 [RRS15b].

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- Multi-constraint percentile queries for LTL [EKVY08].
  - Closest to our work.
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- Multi-constraint percentile queries for LTL [EKVY08].
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- Recent work on percentile queries + $\mathbb{E}$ for MP [CKK15].

Summary

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Thank you! Any question?

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$f \in \mathcal{F}$	P [CH09]	P	P(M)·E($\mathcal{Q}$) PSPACE-h.
$\overline{\text{MP}}$	P [Put94]	P	P
$\underline{\text{MP}}$	P [Put94]	P(M)·E($\mathcal{Q}$)	P(M)·E($\mathcal{Q}$)
SP	P(M)·P$_{ps}$($\mathcal{Q}$) [HK15] PSPACE-h. [HK15]	P(M)·P$_{ps}$($\mathcal{Q}$) (one target) PSPACE-h. [HK15]	P(M)·E($\mathcal{Q}$) PSPACE-h. [HK15]
ε-gap DS	P$_{ps}$($M, \mathcal{Q}, \varepsilon$) NP-h.	P$_{ps}$(M, ε)·E($\mathcal{Q}$) NP-h.	P$_{ps}$(M, ε)·E($\mathcal{Q}$) PSPACE-h.

In most cases, only **polynomial in the model size**.

▷ In practice, the query size can often be bounded while the model can be very large.

Results overview: sketches

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	P(M)·E($\mathcal{Q}$) [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	P(M)·E($\mathcal{Q}$) PSPACE-h.
$\overline{\text{MP}}$	P [Put94]	P	P
$\underline{\text{MP}}$	P [Put94]	P(M)·E($\mathcal{Q}$)	P(M)·E($\mathcal{Q}$)
SP	P(M)·P_{ps}($\mathcal{Q}$) [HK15] PSPACE-h. [HK15]	P(M)·P_{ps}($\mathcal{Q}$) (one target) PSPACE-h. [HK15]	P(M)·E($\mathcal{Q}$) PSPACE-h. [HK15]
ε-gap DS	$P_{ps}(M, \mathcal{Q}, \varepsilon)$ NP-h.	$P_{ps}(M, \varepsilon)$·E($\mathcal{Q}$) NP-h.	$P_{ps}(M, \varepsilon)$·E($\mathcal{Q}$) PSPACE-h.

No time to discuss every result!

Results overview: sketches

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	$P(M)\cdot E(\mathcal{Q})$ [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	$P(M)\cdot E(\mathcal{Q})$ PSPACE-h.
$\overline{MP}$	P [Put94]	P	P
$\underline{MP}$	P [Put94]	$P(M)\cdot E(\mathcal{Q})$	$P(M)\cdot E(\mathcal{Q})$
SP	$P(M)\cdot P_{ps}(\mathcal{Q})$ [HK15] PSPACE-h. [HK15]	$P(M)\cdot P_{ps}(\mathcal{Q})$ (one target) PSPACE-h. [HK15]	$P(M)\cdot E(\mathcal{Q})$ PSPACE-h. [HK15]
ε-gap DS	$P_{ps}(M, \mathcal{Q}, \varepsilon)$ NP-h.	$P_{ps}(M, \varepsilon)\cdot E(\mathcal{Q})$ NP-h.	$P_{ps}(M, \varepsilon)\cdot E(\mathcal{Q})$ PSPACE-h.

Four groups of results

1. **Reachability**. Algorithm based on multi-objective linear programming (LP) in [EKVY08]. We refine the complexity analysis, provide LBs and tractable subclasses.
   - ▷ **Useful tool** for many payoff functions!

Results overview: sketches

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	$P(M)\cdot E(\mathcal{Q})$ [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	$P(M)\cdot E(\mathcal{Q})$ PSPACE-h.
$\overline{\text{MP}}$	P [Put94]	P	P
$\underline{\text{MP}}$	P [Put94]	$P(M)\cdot E(\mathcal{Q})$	$P(M)\cdot E(\mathcal{Q})$
SP	$P(M)\cdot P_{ps}(\mathcal{Q})$ [HK15] PSPACE-h. [HK15]	$P(M)\cdot P_{ps}(\mathcal{Q})$ (one target) PSPACE-h. [HK15]	$P(M)\cdot E(\mathcal{Q})$ PSPACE-h. [HK15]
ε-gap DS	$P_{ps}(M, \mathcal{Q}, \varepsilon)$ NP-h.	$P_{ps}(M, \varepsilon)\cdot E(\mathcal{Q})$ NP-h.	$P_{ps}(M, \varepsilon)\cdot E(\mathcal{Q})$ PSPACE-h.

Four groups of results

2 $\mathcal{F}$ **and** $\overline{\text{MP}}$. Easiest cases.

- inf and sup: reduction to *multiple reachability*.
- lim inf, lim sup and $\overline{\text{MP}}$: *maximal end-component* (MEC) decomposition + reduction to multiple reachability.

Results overview: sketches

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	$\mathsf{P}(\mathcal{M})\cdot\mathsf{E}(\mathcal{Q})$ [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	$\mathsf{P}(\mathcal{M})\cdot\mathsf{E}(\mathcal{Q})$ PSPACE-h.
$\overline{\mathsf{MP}}$	P [Put94]	P	P
$\underline{\mathsf{MP}}$	P [Put94]	$\mathsf{P}(\mathcal{M})\cdot\mathsf{E}(\mathcal{Q})$	$\mathsf{P}(\mathcal{M})\cdot\mathsf{E}(\mathcal{Q})$
SP	$\mathsf{P}(\mathcal{M})\cdot\mathsf{P}_{ps}(\mathcal{Q})$ [HK15] PSPACE-h. [HK15]	$\mathsf{P}(\mathcal{M})\cdot\mathsf{P}_{ps}(\mathcal{Q})$ (one target) PSPACE-h. [HK15]	$\mathsf{P}(\mathcal{M})\cdot\mathsf{E}(\mathcal{Q})$ PSPACE-h. [HK15]
ε-gap DS	$\mathsf{P}_{ps}(\mathcal{M}, \mathcal{Q}, \varepsilon)$ NP-h.	$\mathsf{P}_{ps}(\mathcal{M}, \varepsilon)\cdot\mathsf{E}(\mathcal{Q})$ NP-h.	$\mathsf{P}_{ps}(\mathcal{M}, \varepsilon)\cdot\mathsf{E}(\mathcal{Q})$ PSPACE-h.

Four groups of results

3. $\underline{\mathsf{MP}}$. Technically involved.
   - ▷ Inside MECs: (a) strategies satisfying *maximal subsets of constraints*, (b) combine them linearly.
   - ▷ Overall: write an LP combining multiple reachability toward MECs and those linear combinations equations.

Results overview: sketches

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	$\mathsf{P}(M)\cdot\mathsf{E}(\mathcal{Q})$ [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	$\mathsf{P}(M)\cdot\mathsf{E}(\mathcal{Q})$ PSPACE-h.
$\overline{\text{MP}}$	P [Put94]	P	P
$\underline{\text{MP}}$	P [Put94]	$\mathsf{P}(M)\cdot\mathsf{E}(\mathcal{Q})$	$\mathsf{P}(M)\cdot\mathsf{E}(\mathcal{Q})$
SP	$\mathsf{P}(M)\cdot\mathsf{P}_{ps}(\mathcal{Q})$ [HK15] PSPACE-h. [HK15]	$\mathsf{P}(M)\cdot\mathsf{P}_{ps}(\mathcal{Q})$ (one target) PSPACE-h. [HK15]	$\mathsf{P}(M)\cdot\mathsf{E}(\mathcal{Q})$ PSPACE-h. [HK15]
ε-gap DS	$\mathsf{P}_{ps}(M, \mathcal{Q}, \varepsilon)$ NP-h.	$\mathsf{P}_{ps}(M, \varepsilon)\cdot\mathsf{E}(\mathcal{Q})$ NP-h.	$\mathsf{P}_{ps}(M, \varepsilon)\cdot\mathsf{E}(\mathcal{Q})$ PSPACE-h.

Four groups of results

4. **SP and DS**. Based on *unfoldings* and multiple reachability.
   - ▷ Need finite and bounded unfoldings.
   - ▷ For SP, we bound the size of the unfolding by *node merging*.
   - ▷ For DS, we can only *approximate* the answer in general. Need to analyze the cumulative error due to necessary *roundings*.