Percentile Queries in Multi-Dimensional Markov Decision Processes

Mickael Randour (LSV - CNRS & ENS Cachan)
Jean-François Raskin (ULB) Ocan Sankur (ULB)

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LACL seminar, UPEC

The talk in one slide

Strategy synthesis for Markov Decision Processes (MDPs)

Finding **good** controllers for systems interacting with a *stochastic* environment.

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- Good? Performance evaluated through *payoff functions*.
- Usual problem is to optimize the *expected performance* or the *probability of achieving a given performance level*.
- Not sufficient for many practical applications.
  - ▷ Several extensions, more expressive but also more complex...

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Aim of this talk

Multi-constraint percentile queries: generalizes the problem to multiple dimensions, multiple constraints.

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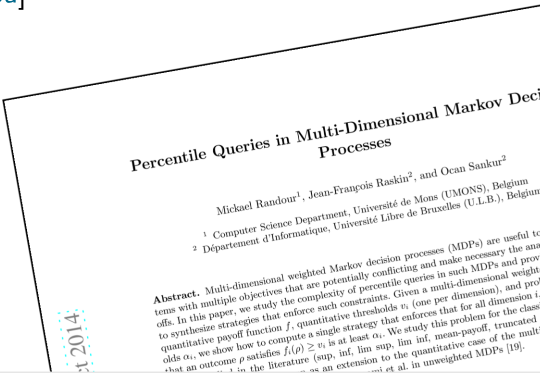
Full paper available on arXiv [RRS14]: abs/1410.4801

To appear in CAV'15 [RRS15a]



**Percentile Queries in Multi-Dimensional Markov Deci[...]
Processes**

Mickael Randour[1], Jean-François Raskin[2], and Ocan Sankur[2]

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Abstract. Multi-dimensional weighted Markov decision processes (MDPs) are useful to [...]
tems with multiple objectives that are potentially conflicting and make necessary the ana[...]
offs. In this paper, we study the complexity of percentile queries in such MDPs and prov[...]
to synthesize strategies that enforce such constraints. Given a multi-dimensional weight[...]
quantitative payoff function f, quantitative thresholds v_i (one per dimension), and pro[...]
olds α_i, we show how to compute a single strategy that enforces that for all dimension i, [...]
that an outcome ρ satisfies $f_i(\rho) \geq v_i$ is at least α_i. We study this problem for the class[...]
[...] in the literature (sup, inf, lim sup, lim inf, mean-payoff, truncated [...]
[...] as an extension to the quantitative case of the multi[...]
[...] et al. in unweighted MDPs [19].

Context

- Verification and synthesis:
  - a reactive **system** to *control*,
  - an *interacting* **environment**,
  - a **specification** to *enforce*.

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  - ▷ Antagonistic environment: 2-player game on graph.
  - ▷ **Stochastic environment: MDP**.

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  - ▷ **Stochastic environment: MDP**.
- **Quantitative** specifications. Examples:
  - ▷ Reach a state s before x time units $\leadsto$ shortest path.
  - ▷ Minimize the average response-time $\leadsto$ mean-payoff.

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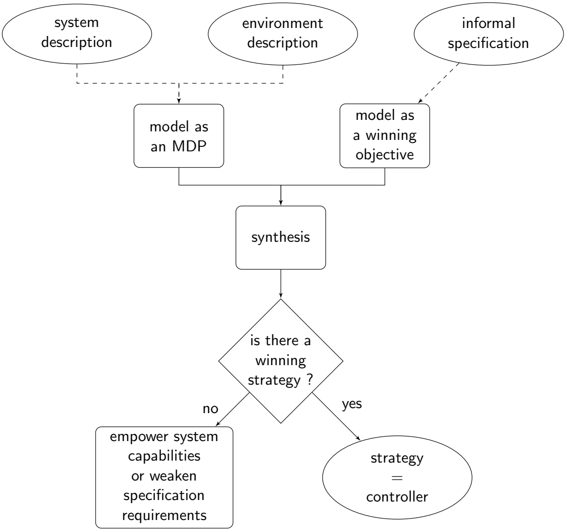
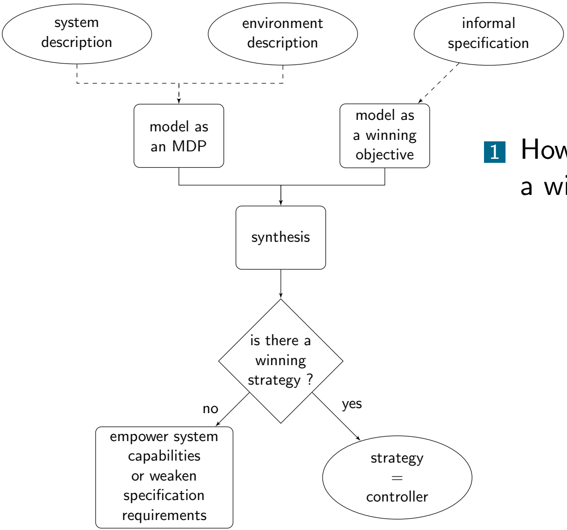
- **Quantitative** specifications. Examples:
  - ▷ Reach a state s before x time units $\rightsquigarrow$ shortest path.
  - ▷ Minimize the average response-time $\rightsquigarrow$ mean-payoff.
- Focus on **multi-criteria quantitative models**
  - ▷ to reason about *trade-offs* and *interplays*.

Strategy (policy) synthesis for MDPs

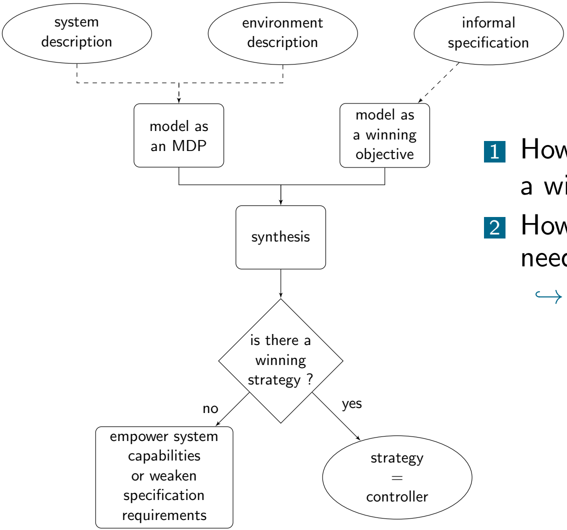


Strategy (policy) synthesis for MDPs



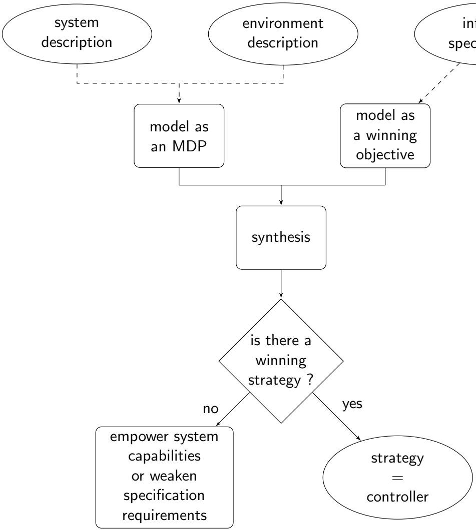
1. How complex is it to decide if a winning strategy exists?

Strategy (policy) synthesis for MDPs



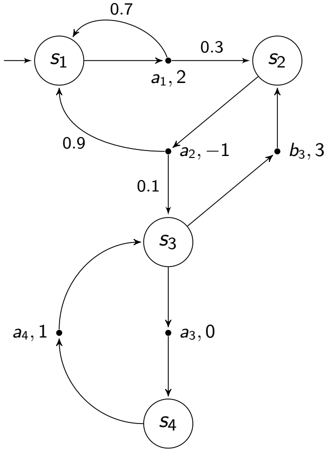
1. How complex is it to decide if a winning strategy exists?
2. How complex such a strategy needs to be?
   ↪ **Simpler is better.**

Strategy (policy) synthesis for MDPs



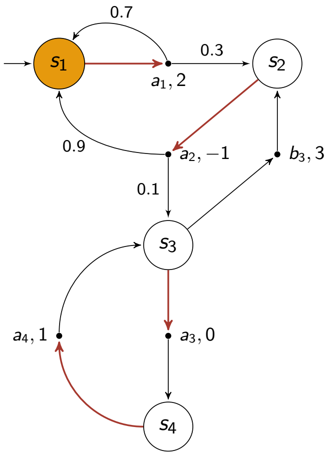
1. How complex is it to decide if a winning strategy exists?
2. How complex such a strategy needs to be?
   $\hookrightarrow$ **Simpler is better.**
3. Can we synthesize one efficiently?

Markov decision processes



- **MDP** $M = (S, A, \delta, w)$
  - ▷ finite sets of states S and actions A
  - ▷ probabilistic transition $\delta\colon S \times A \to \mathcal{D}(S)$
  - ▷ weight function $w\colon A \to \mathbb{Z}^d$
- **Run** (or play): $\rho = s_1 a_1 \dots a_{n-1} s_n \dots$ such that $\delta(s_i, a_i, s_{i+1}) > 0$ for all $i \geq 1$
  - ▷ set of runs $\mathcal{R}(M)$
  - ▷ set of histories (finite runs) $\mathcal{H}(M)$
- **Strategy** $\sigma\colon \mathcal{H}(M) \to \mathcal{D}(A)$
  - ▷ $\forall\, h$ ending in s, $\text{Supp}(\sigma(h)) \in A(s)$

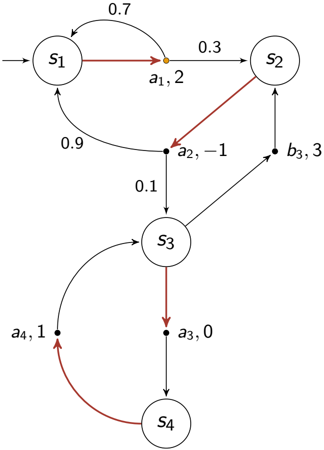
Markov decision processes



Sample *pure memoryless* strategy σ

Sample run $\rho = s_1$

Markov decision processes



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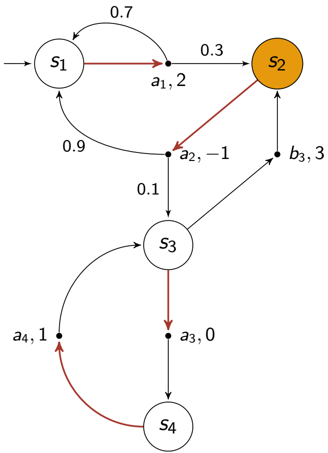
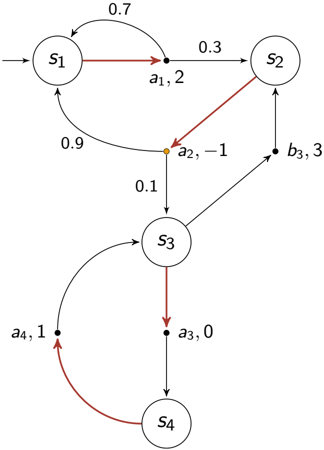


Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2$

Markov decision processes



Sample *pure memoryless* strategy σ

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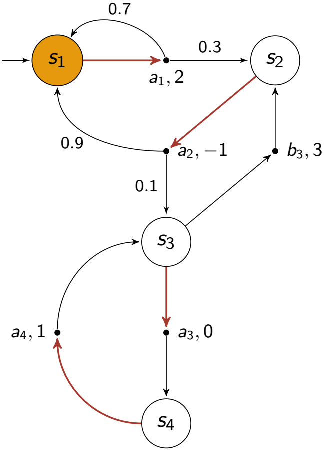
Markov decision processes



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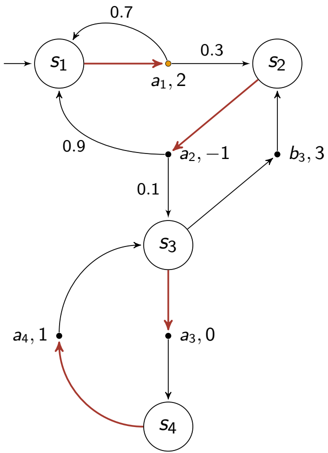
Markov decision processes



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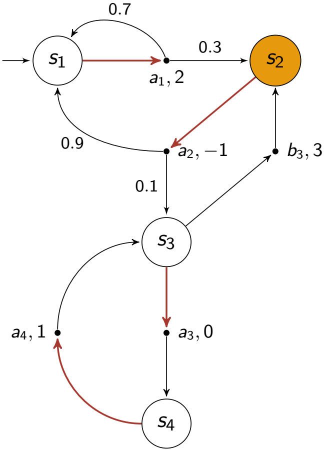

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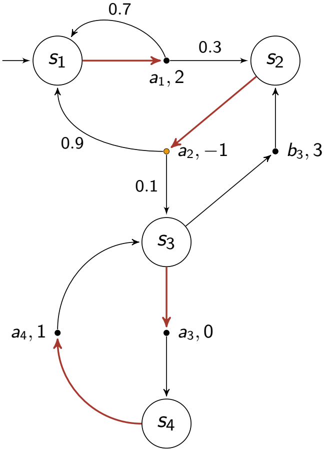

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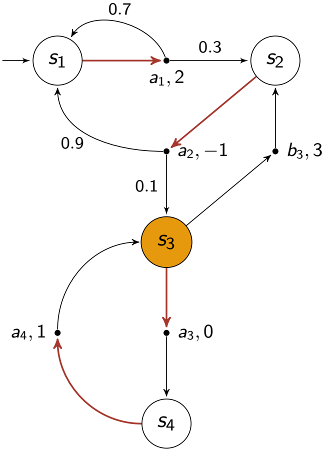
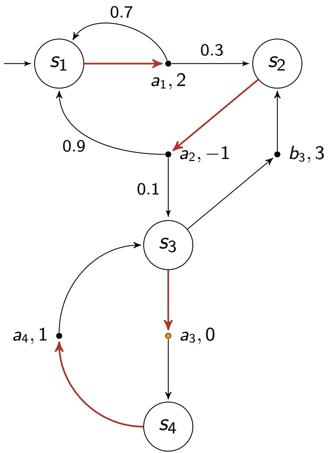


Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2 a_2 s_1 a_1 s_2 a_2 s_3$

Markov decision processes



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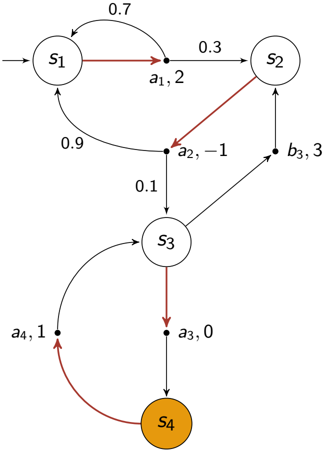

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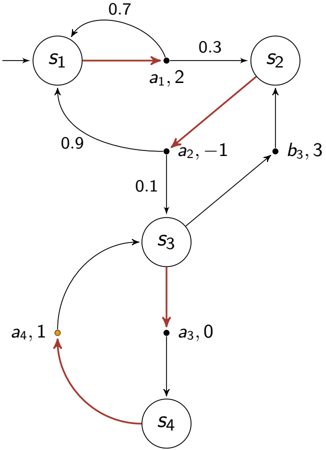
Sample run $\rho = s_1 a_1 s_2 a_2 s_1 a_1 s_2 a_2 s_3 a_3 s_4$

Markov decision processes



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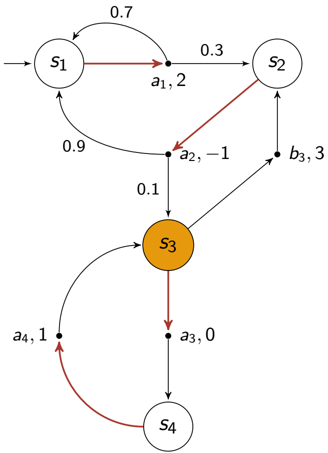
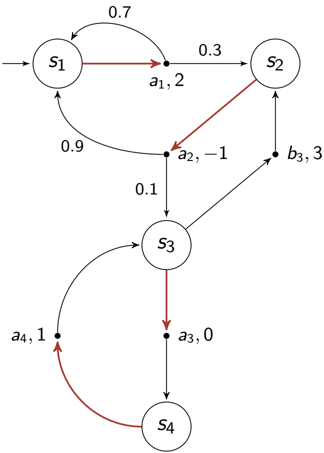


Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2 a_2 s_1 a_1 s_2 a_2 (s_3 a_3 s_4 a_4)^\omega$

Markov decision processes

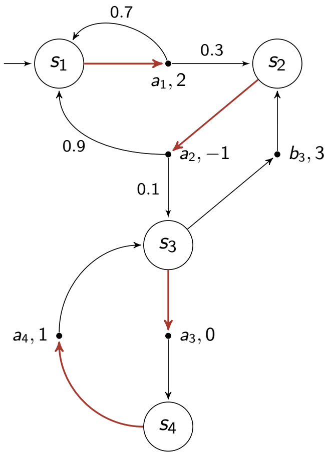


Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2 a_2 s_1 a_1 s_2 a_2 (s_3 a_3 s_4 a_4)^\omega$

Other possible run $\rho' = s_1 a_1 s_2 a_2 (s_3 a_3 s_4 a_4)^\omega$

Markov decision processes



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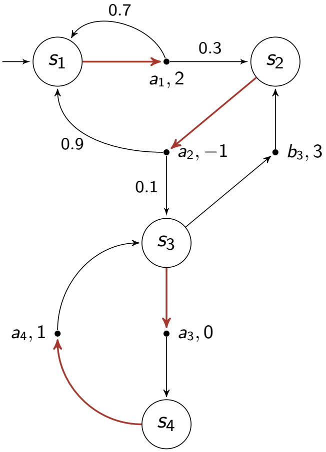
- Strategies may use
  - finite or infinite **memory**
  - **randomness**
- **Payoff functions** map runs to numerical values
  - truncated sum up to $T = \{s_3\}$: $\mathsf{TS}^T(\rho) = 2$, $\mathsf{TS}^T(\rho') = 1$
  - mean-payoff: $\underline{\mathsf{MP}}(\rho) = \underline{\mathsf{MP}}(\rho') = 1/2$
  - many more

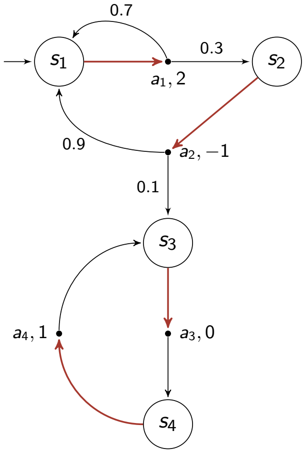
Markov chains



Once initial state s_{init} and strategy σ fixed, fully stochastic process

$\rightsquigarrow$ **Markov chain (MC)**

Markov chains



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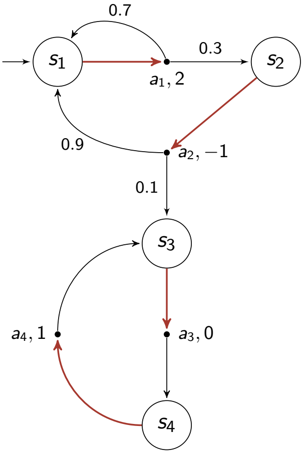
$\rightsquigarrow$ **Markov chain** (**MC**)

State space = product of the MDP and the memory of σ

Markov chains



Once initial state s_{init} and strategy σ fixed, fully stochastic process

$\rightsquigarrow$ **Markov chain** (**MC**)

State space = product of the MDP and the memory of σ

- Event $\mathcal{E} \subseteq \mathcal{R}(M)$
  - ▷ probability $\mathbb{P}^{\sigma}_{M,s_{\text{init}}}(\mathcal{E})$
- Measurable $f\colon \mathcal{R}(M) \to (\mathbb{R} \cup \{-\infty, \infty\})^d$
  - ▷ expected value $\mathbb{E}^{\sigma}_{M,s_{\text{init}}}(f)$

Single-constraint percentile problem

Ensuring a given performance level with sufficient probability

- uni-dimensional weight function $w\colon A \to \mathbb{Z}$ and payoff function $f\colon \mathcal{R}(M) \to \mathbb{R} \cup \{-\infty, \infty\}$
- well-studied for various payoffs

Single-constraint percentile problem

Given MDP $M = (S, A, \delta, w)$, initial state s_{init}, payoff function f, value threshold $v \in \mathbb{Q}$, and probability threshold $\alpha \in [0,1] \cap \mathbb{Q}$, decide if there exists a strategy σ such that

$$\mathbb{P}^{\sigma}_{M,s_{\mathsf{init}}}\left[\{\rho \in \mathcal{R}_{s_{\mathsf{init}}}(M) \mid f(\rho) \geq v\}\right] \geq \alpha.$$

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- **percentile constraint**, shortened $\mathbb{P}^{\sigma}_{M,s_{\mathsf{init}}}\left[f \geq v\right] \geq \alpha$

Illustration: stochastic shortest path problem

Shortest path (SP) problem for *weighted graphs*

Given state $s \in S$ and target set $T \subseteq S$, find a path from s to a state $t \in T$ that *minimizes* the sum of weights along edges.

- PTIME algorithms (Dijkstra, Bellman-Ford, etc) [CGR96]

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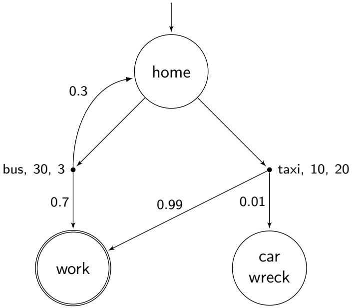
For SP, we focus on MDPs with **positive weights**

- ▷ **Truncated sum** payoff function for $\rho = s_1 a_1 s_2 a_2 \dots$ and target set T:

$$\mathsf{TS}^T(\rho) = \begin{cases} \sum_{j=1}^{n-1} w(a_j) \text{ if } s_n \text{ first visit of } T \\ \infty \text{ if } T \text{ is never reached} \end{cases}$$

Illustration: stochastic shortest path problem



Two-dimensional weights on actions: *time* and *cost*.

Often necessary to consider trade-offs: e.g., between the probability to reach work in due time and the risks of an expensive journey.

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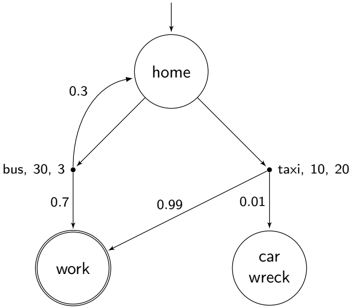

Classical problem considers only a **single percentile constraint**.

- **C1**: 80% of runs reach work in at most 40 minutes.
  - ▷ Taxi $\rightsquigarrow \leq 10$ minutes with probability $0.99 > 0.8$.

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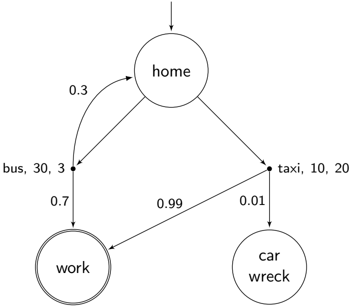

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- **C2**: 50% of them cost at most 10$ to reach work.
  - ▷ Bus $\rightsquigarrow \geq 70\%$ of the runs reach work for 3$.

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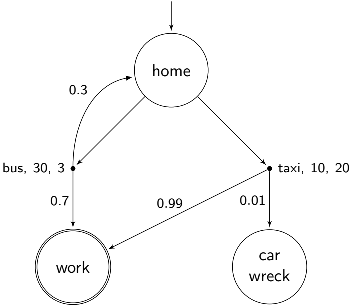

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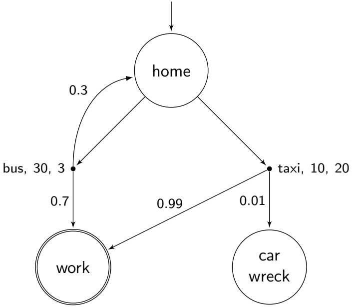
Taxi $\not\models$ C2, bus $\not\models$ C1. What if we want C1 $\wedge$ C2?

Illustration: stochastic shortest path problem

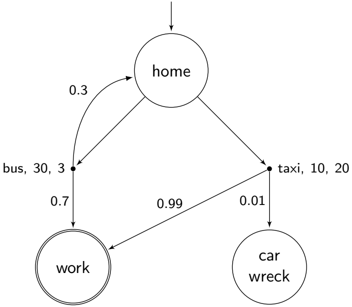


- **C1**: 80% of runs reach work in at most 40 minutes.
- **C2**: 50% of them cost at most 10$ to reach work.

Study of **multi-constraint percentile queries**.

- ▷ Sample strategy: bus once, then taxi. Requires *memory*.
- ▷ Another strategy: bus with probability 3/5, taxi with probability 2/5. Requires *randomness*.

Illustration: stochastic shortest path problem



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Study of **multi-constraint percentile queries**.

In general, *both* memory *and* randomness are required.

$\neq$ classical problems (single constraint, expected value, etc)

Multi-constraint percentile problem

Multi-constraint percentile problem

Given d-dimensional MDP $M = (S, A, \delta, w)$, initial state s_{init}, payoff function f, and $q \in \mathbb{N}$ percentile constraints described by dimensions $l_i \in \{1, \ldots, d\}$, value thresholds $v_i \in \mathbb{Q}$ and probability thresholds $\alpha_i \in [0, 1] \cap \mathbb{Q}$, where $i \in \{1, \ldots, q\}$, decide if there exists a strategy σ such that query $\mathcal{Q}$ holds, with

$$\mathcal{Q} := \bigwedge_{i=1}^{q} \mathbb{P}^{\sigma}_{M,s_{\text{init}}}\left[f_{l_i} \geq v_i\right] \geq \alpha_i.$$

Very general framework allowing for: multiple constraints related to $\neq$ or $=$ dimensions, $\neq$ value and probability thresholds.

- $\leadsto$ For SP, even $\neq$ targets for each constraint.
- $\leadsto$ Great flexibility in modeling applications.

Results overview (1/2)

- **Wide range of payoff functions**
  - ▷ multiple reachability,
  - ▷ inf, sup, lim inf, lim sup,
  - ▷ mean-payoff ($\overline{\mathsf{MP}}$, $\underline{\mathsf{MP}}$),
  - ▷ shortest path (SP),
  - ▷ discounted sum (DS).

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- **Several variants**:
  - multi-dim. multi-constraint,
  - single-dim. multi-constraint,
  - single-constraint.

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- **Wide range of payoff functions**
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  - shortest path (SP),
  - discounted sum (DS).
- **Several variants**:
  - multi-dim. multi-constraint,
  - single-dim. multi-constraint,
  - single-constraint.
- For each one:
  - algorithms,
  - lower bounds,
  - memory requirements.

$\rightsquigarrow$ **Complete picture** for this new framework.

Results overview (2/2)

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	$P(M) \cdot E(\mathcal{Q})$ [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	$P(M) \cdot E(\mathcal{Q})$ PSPACE-h.
$\overline{\mathsf{MP}}$	P [Put94]	P	P
$\underline{\mathsf{MP}}$	P [Put94]	$P(M) \cdot E(\mathcal{Q})$	$P(M) \cdot E(\mathcal{Q})$
SP	$P(M) \cdot P_{ps}(\mathcal{Q})$ [HK15b] PSPACE-h. [HK15b]	$P(M) \cdot P_{ps}(\mathcal{Q})$ (one target) PSPACE-h. [HK15b]	$P(M) \cdot E(\mathcal{Q})$ PSPACE-h. [HK15b]
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- ▷ $\mathcal{F} = \{\inf, \sup, \liminf, \limsup\}$
- ▷ M = model size, $\mathcal{Q}$ = query size
- ▷ $P(x)$, $E(x)$ and $P_{ps}(x)$ resp. denote polynomial, exponential and pseudo-polynomial time in parameter x.

All results without reference are new.

Results overview (2/2)

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	P(M)·E($\mathcal{Q}$) [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	P(M)·E($\mathcal{Q}$) PSPACE-h.
$\overline{\text{MP}}$	P [Put94]	P	P
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ε-gap DS	P$_{ps}$($M, \mathcal{Q}, \varepsilon$) NP-h.	P$_{ps}$(M, ε)·E($\mathcal{Q}$) NP-h.	P$_{ps}$(M, ε)·E($\mathcal{Q}$) PSPACE-h.

In most cases, only **polynomial in the model size**.

- ▷ In practice, the query size can often be bounded while the model can be very large.

Results overview (2/2)

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
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No time to discuss every result!

Results overview (2/2)

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Four groups of results

1. **Reachability**. Algorithm based on multi-objective linear programming (LP) in [EKVY08]. We refine the complexity analysis, provide LBs and tractable subclasses.
   - ▷ **Useful tool** for many payoff functions!

Results overview (2/2)

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
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Four groups of results

2 $\mathcal{F}$ **and** $\overline{\mathsf{MP}}$. Easiest cases.

- inf and sup: reduction to *multiple reachability*.
- lim inf, lim sup and $\overline{\mathsf{MP}}$: *maximal end-component* (MEC) decomposition + reduction to multiple reachability.

Results overview (2/2)

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Four groups of results

3. $\underline{\mathsf{MP}}$. Technically involved.
   - Inside MECs: (a) strategies satisfying *maximal subsets of constraints*, (b) combine them linearly.
   - Overall: write an LP combining multiple reachability toward MECs and those linear combinations equations.

Results overview (2/2)

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Four groups of results

4. **SP and DS**. Based on *unfoldings* and multiple reachability.
   - ▷ For SP, we bound the size of the unfolding by *node merging*.
   - ▷ For DS, we can only *approximate* the answer in general. Need to analyze the cumulative error due to necessary *roundings*.

Results overview (2/2)

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Four groups of results

4. **SP and DS**.
   - $\rightsquigarrow$ **Technical focus of this talk.**
     - ▷ Intuitive unfoldings, interesting tricks for DS.
     - ▷ Start simple and iteratively extend the solution.

Some related work

- **Same philosophy** (i.e., beyond uni-dimensional $\mathbb{E}$ or $\mathbb{P}$ maximization), $\neq$ approaches.
  - ▷ Beyond worst-case synthesis: $\mathbb{E}$ + worst-case [BFRR14b].
  - ▷ Survey of recent extensions in VMCAI'15 [RRS15b].

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- Multi-dim. MDPs: DS [CMH06], MP [BBC+14, FKR95].

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  - ▷ All with a *single* constraint.
- Multi-constraint percentile queries for LTL [EKVY08].
  - ▷ Closest to our work.
  - ▷ We use *multiple reachability*.

Single-constraint queries

Single-constraint percentile problem for SP

Given MDP $M = (S, A, \delta, w)$, initial state s_{init}, target set T, threshold $v \in \mathbb{N}$, and probability threshold $\alpha \in [0,1] \cap \mathbb{Q}$, decide if there exists a strategy σ such that $\mathbb{P}^{\sigma}_{M,s_{\mathsf{init}}}\left[\mathsf{TS}^{T} \leq v\right] \geq \alpha$.

- Hypothesis: all weights are non-negative.

Theorem

The above problem can be decided in pseudo-polynomial time and is PSPACE-hard. Optimal pure strategies with pseudo-polynomial memory exist and can be constructed in pseudo-polynomial time.

- Polynomial in the size of the MDP, but also in the threshold v.
- See [HK15b] for hardness.

Pseudo-PTIME algorithm (1/2)

Key idea: pseudo-PTIME reduction to the **stochastic reachability problem** (**SR** - single target).

Pseudo-PTIME algorithm (1/2)

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SR problem

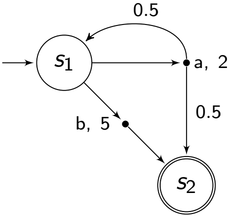
Given unweighted MDP $M = (S, A, \delta)$, initial state s_{init}, target set T and probability threshold $\alpha \in [0, 1] \cap \mathbb{Q}$, decide if there exists a strategy σ such that $\mathbb{P}^{\sigma}_{M, s_{\mathsf{init}}}\left[\Diamond T\right] \geq \alpha$.

Theorem

The SR problem can be decided in polynomial time. Optimal pure memoryless strategies exist and can be constructed in polynomial time.

- Linear programming.

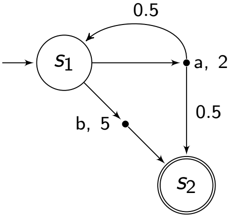
Pseudo-PTIME algorithm (2/2)



Sketch of the reduction

1. Start from M, $T = \{s_2\}$, and $v = 7$.

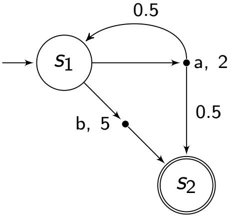
Pseudo-PTIME algorithm (2/2)

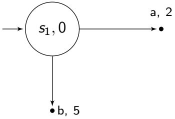


Sketch of the reduction

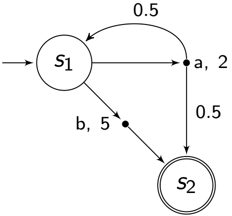
1. Start from M, $T = \{s_2\}$, and $v = 7$.
2. Build M_v by unfolding M, tracking the current sum *up to the threshold* v, and integrating it in the states of the expanded MDP.

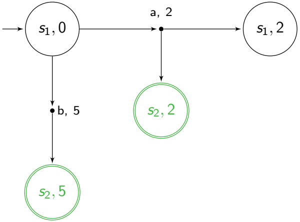
Pseudo-PTIME algorithm (2/2)





Pseudo-PTIME algorithm (2/2)





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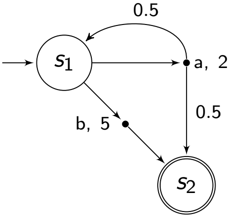
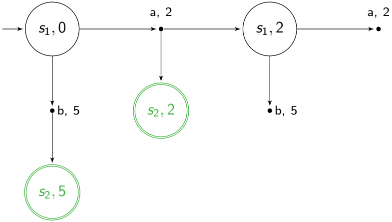
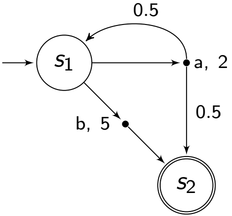
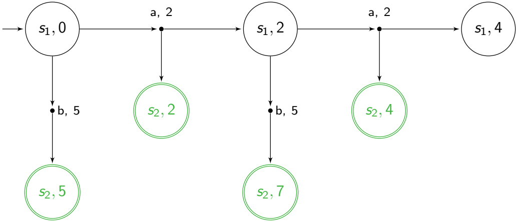


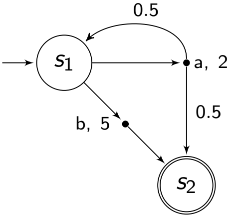


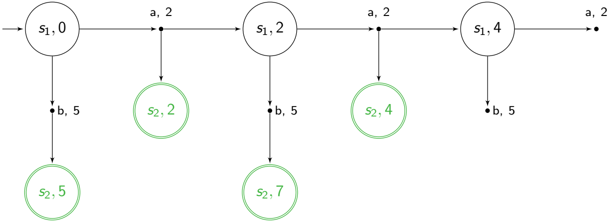
Pseudo-PTIME algorithm (2/2)





Pseudo-PTIME algorithm (2/2)





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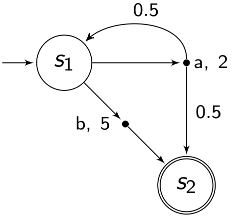

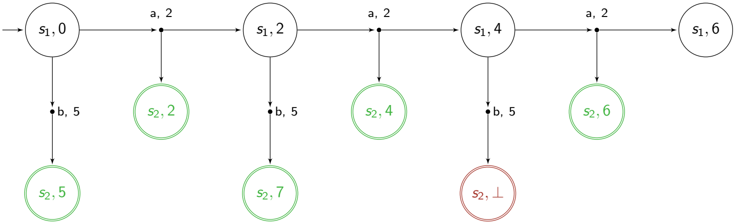

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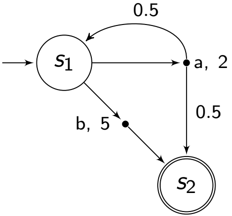
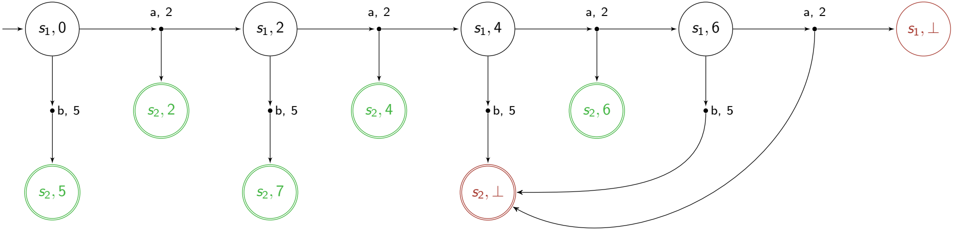
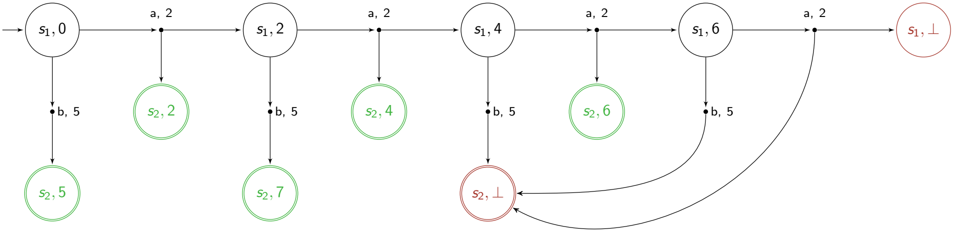




Pseudo-PTIME algorithm (2/2)

3 Bijection between runs of M and M_v

$$\mathsf{TS}^T(\rho) \leq v \quad \Leftrightarrow \quad \rho' \models \Diamond T',\ T' = T \times \{0, 1, \ldots, v\}$$

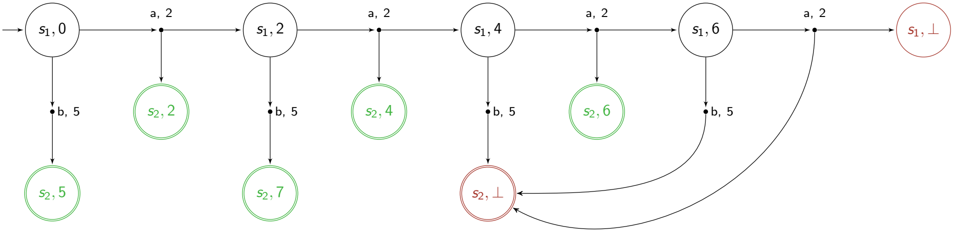


Pseudo-PTIME algorithm (2/2)

3. Bijection between runs of M and M_v

$$\mathsf{TS}^T(\rho) \leq v \quad \Leftrightarrow \quad \rho' \models \Diamond T',\ T' = T \times \{0, 1, \ldots, v\}$$

4. Solve the SR problem on M_v
   - Memoryless strategy in $M_v \rightsquigarrow$ pseudo-polynomial memory in M in general



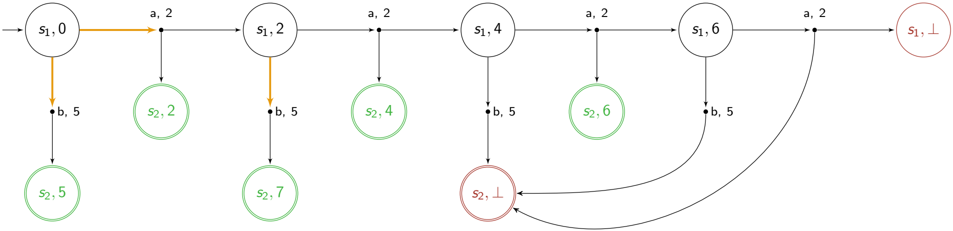
Pseudo-PTIME algorithm (2/2)

If we just want to minimize the risk of exceeding $v = 7$,

- ▷ an obvious possibility is to play b directly,
- ▷ playing a only once is also acceptable.

For the single-constraint problem, **both strategies are equivalent**

$\rightsquigarrow$ we can discriminate them with richer queries



Multi-constraint queries (1/2)

Multi-constraint percentile problem for SP

Given d-dimensional MDP $M = (S, A, \delta, w)$, initial state s_{init} and $q \in \mathbb{N}$ percentile constraints described by target sets $T_i \subseteq S$, dimensions $l_i \in \{1, \ldots, d\}$, value thresholds $v_i \in \mathbb{N}$ and probability thresholds $\alpha_i \in [0, 1] \cap \mathbb{Q}$, where $i \in \{1, \ldots, q\}$, decide if there exists a strategy σ such that query $\mathcal{Q}$ holds, with

$$\mathcal{Q} := \bigwedge_{i=1}^{q} \mathbb{P}^{\sigma}_{M, s_{\mathsf{init}}}\left[\mathsf{TS}^{T_i}_{l_i} \leq v_i\right] \geq \alpha_i,$$

where $\mathsf{TS}^{T_i}_{l_i}$ denotes the truncated sum on dimension l_i and w.r.t. target set T_i.

Multi-constraint queries (2/2)

Theorem

This problem can be decided in

- exponential time in general,
- pseudo-polynomial time for single-dimension single-target multi-contraint queries.

It is PSPACE-hard even for single-constraint queries. Randomized exponential-memory strategies are always sufficient and in general necessary, and can be constructed in exponential time.

▷ Polynomial in the size of the MDP, blowup due to the query.

▷ Hardness already true for single-constraint [HK15b].

$\rightsquigarrow$ wide extension for basically no price in complexity.

⚠ Undecidable for arbitrary weights (2CM reduction)!

EXPTIME / pseudo-PTIME algorithm

1. Build an unfolded MDP M_v similar to single-constraint case:
   - ▷ stop unfolding when *all* dimensions reach sum $v = \max_i v_i$.

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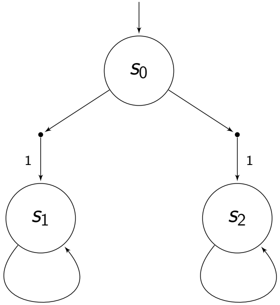
   - $\rho \models$ constraint i in $M \Leftrightarrow \rho' \models \Diamond R_i$ in M_v.
4. Solve a **multiple reachability problem** on M_v.
   - Generalizes the SR problem [EKVY08, RRS14].
   - Time polynomial in M_v but exponential in q.
   - Single-dim. single target queries $\Rightarrow$ absorbing targets $\Rightarrow$ polynomial-time algorithm for multiple reachability.

Randomness is always necessary

- ▷ For any payoff function and a sufficiently general query.
- ▷ Example: multiple reachability.

$$\exists ?\, \sigma : \ \mathbb{P}^{\sigma}_{M,s_0}\left[\Diamond s_1\right] \geq 0.5 \ \wedge \ \mathbb{P}^{\sigma}_{M,s_0}\left[\Diamond s_2\right] \geq 0.5$$



Need to play s_1 and s_2 with probability $1/2$.

Multi-constraint queries

Multi-constraint percentile problem for DS

Given *d*-dimensional MDP $M = (S, A, \delta, w)$, initial state s_{init} and $q \in \mathbb{N}$ percentile constraints described by discount factors $\lambda_i \in]0, 1[\cap \mathbb{Q}$, dimensions $l_i \in \{1, \ldots, d\}$, value thresholds $v_i \in \mathbb{N}$ and probability thresholds $\alpha_i \in [0, 1] \cap \mathbb{Q}$, where $i \in \{1, \ldots, q\}$, decide if there exists a strategy σ such that query $\mathcal{Q}$ holds, with

$$\mathcal{Q} := \bigwedge_{i=1}^{q} \mathbb{P}^{\sigma}_{M,s_{\text{init}}}\left[\mathsf{DS}^{\lambda_i}_{l_i} \geq v_i\right] \geq \alpha_i,$$

where $\mathsf{DS}^{\lambda_i}_{l_i}(\rho) = \sum_{j=1}^{\infty} \lambda_i^j \cdot w_{l_i}(a_j)$ denotes the discounted sum on dimension l_i and w.r.t. discount factor λ_i.

We allow **arbitrary** weights.

Precise discounted sum problem is hard

Precise DS problem

Given value $t \in \mathbb{Q}$, and discount factor $\lambda \in]0, 1[$, does there exist an infinite binary sequence $\tau = \tau_1\tau_2\tau_3 \ldots \in \{0, 1\}^\omega$ such that $\sum_{j=1}^{\infty} \lambda^j \cdot \tau_j = t$?

- Reduces to an almost-sure percentile problem on a single-state 2-dim. MDP.
- **Still not known to be decidable!**
  - $\rightsquigarrow$ related to open questions such as the *universality problem for discounted-sum automata* [BHO15, CFW13, BH14].

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  - $\leadsto$ related to open questions such as the *universality problem for discounted-sum automata* [BHO15, CFW13, BH14].

We cannot solve the exact problem but we can approximate correct answers.

ε-gap percentile problem (1/3)

- Classical decision problem.
  - Two types of inputs: *yes*-inputs and *no*-inputs.
  - Correct answers required for both types.

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  - ▷ Three types: *yes*-inputs, *no*-inputs, *remaining* inputs.
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- ε-gap problem.
  - ▷ The uncertainty zone can be made **arbitrarily small**, parametrized by value $\varepsilon > 0$.

ε-gap percentile problem (2/3)

We build an **algorithm**.

- Inputs: query $\mathcal{Q}$ and precision factor $\varepsilon > 0$.
- Output: Yes, No or Unknown.
  - ▷ If Yes, then a strategy exists and can be synthesized.
  - ▷ If No, then no strategy exists.
  - ▷ Answer Unknown can only be output within an uncertainty zone of size $\sim \varepsilon$.
    - $\Rightarrow$ **Incremental approximation scheme.**

ε-gap percentile problem (3/3)

Theorem

There is an algorithm that, given an MDP, a percentile query $\mathcal{Q}$ for the DS and a precision factor $\varepsilon > 0$, solves the following ε-gap problem in exponential time. It answers

- Yes if **there is** a strategy satisfying query $\mathcal{Q}_{2\cdot\varepsilon}$;
- No if **there is no** strategy satisfying query $\mathcal{Q}_{-2\cdot\varepsilon}$;
- and arbitrarily otherwise.

▷ **Shifted query**: $\mathcal{Q}_x \equiv \mathcal{Q}$ with value thresholds $v_i + x$ (all other things being equal).

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+ PSPACE-hard ($d \geq 2$, subset-sum games [Tra06]), NP-hard for $q = 1$ (K-th largest subset problem [BFRR14b, HK15a]), exponential memory sufficient and necessary.

Algorithm: key ideas

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2. **Finite unfolding?**
   - ▷ Sums not necessarily increasing ($\neq$ SP).
     - ⇒ Not easy to know when to stop.
   - ▷ Use the **discount factor**.
     - ⇒ Weights contribute less and less to the sum along a run.
     - ⇒ The range of possible futures narrows the deeper we go.
     - ⇒ Cutting all branches after a pseudo-polynomial depth changes the overall sum by at most $\varepsilon/2$.

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   - ▷ **2-exponential** unfolding overall!

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2. Pseudo-polynomial depth.
   - ▷ **2-exponential** unfolding overall!
3. **Reduce the overall size?**
   - ▷ No direct merging of nodes (no integer labels, $\neq$ SP), too many possible label values.
   - ▷ Introduce a **rounding** scheme of the numbers involved (inspired by [BCF+13]).
     - ⇒ We bound the error due to cumulated roundings by $\varepsilon/2$.
     - ⇒ Single-exponential width.

Algorithm: key ideas

1. Goal: multiple reachability over appropriate *unfolding*.
2. Pseudo-polynomial depth.
3. Single-exponential width.
4. **Leaf labels are off by at most ε.** Classify each leaf w.r.t. each constraint.
   - $\sim$ Same idea as for SP.
     - $\Rightarrow$ Defining target sets for multiple reachability.
   - ▷ Leaves can be good, bad or uncertain (if too close to threshold).

Algorithm: key ideas

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   - Leaves can be good, bad or uncertain (if too close to threshold).
5. Finally, **two multiple reachability problems** to solve.
   - If OK for good leaves, then answer Yes.
   - If KO for good but OK for uncertain, then answer Unknown.
   - If KO for both, then answer No.

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That solves the ε-gap problem.

Summary

- **Multi-constraint percentile queries**.
  - ▷ Generalizes the classical threshold probability problem.
- Wide range of payoffs: reachability, $\inf$, $\sup$, $\liminf$, $\limsup$, mean-payoff, shortest path, discounted sum.
  - ▷ Various techniques are needed.
- **Complexity usually acceptable**.
  - ▷ Often only polynomial in the model size, while exponential in the query size for the most general variants.

Results overview

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	$P(M)\cdot E(\mathcal{Q})$ [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	$P(M)\cdot E(\mathcal{Q})$ PSPACE-h.
$\overline{\mathsf{MP}}$	P [Put94]	P	P
$\underline{\mathsf{MP}}$	P [Put94]	$P(M)\cdot E(\mathcal{Q})$	$P(M)\cdot E(\mathcal{Q})$
SP	$P(M)\cdot P_{ps}(\mathcal{Q})$ [HK15b] PSPACE-h. [HK15b]	$P(M)\cdot P_{ps}(\mathcal{Q})$ (one target) PSPACE-h. [HK15b]	$P(M)\cdot E(\mathcal{Q})$ PSPACE-h. [HK15b]
ε-gap DS	$P_{ps}(M, \mathcal{Q}, \varepsilon)$ NP-h.	$P_{ps}(M, \varepsilon)\cdot E(\mathcal{Q})$ NP-h.	$P_{ps}(M, \varepsilon)\cdot E(\mathcal{Q})$ PSPACE-h.

- ▷ $\mathcal{F} = \{\inf, \sup, \liminf, \limsup\}$
- ▷ M = model size, $\mathcal{Q}$ = query size
- ▷ $P(x)$, $E(x)$ and $P_{ps}(x)$ resp. denote polynomial, exponential and pseudo-polynomial time in parameter x.

Thank you! Any question?

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Stochastic reachability — LP

For each $s \in S$, one variable x_s.

$$\min \sum_{s \in S} x_s$$

under constraints

$$\begin{aligned}
&x_s = 1 && \forall s \in T, \\
&x_s = 0 && \forall s \in S \text{ which cannot reach } T, \\
&x_s \geq \textstyle\sum_{s' \in S} \delta(s, a, s') \cdot x_{s'} && \forall a \in A(s).
\end{aligned}$$

Optimal solution $\Rightarrow \mathbf{v}_s$ is the maximal probability to reach T that can be achieved from s.

Pure memoryless strategy $\sigma^{\mathbf{v}}$ for all $s \notin T$ that can reach T:

$$\sigma^{\mathbf{v}}(s) = \arg \max_{a \in A(s)} \left[\sum_{s' \in S} \delta(s, a, s') \cdot x_{s'} \right].$$

SP with arbitrary weights: undecidability (1/2)

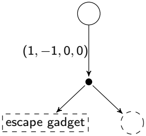
Consider a 2CM $\mathcal{M}$. From this 2CM, we construct an MDP $M = (S, A, \delta, w)$ and a target set of states $T \subset S$, with an initial state $s_{\text{init}} \in S$ such that there exists a strategy $\sigma \in \Sigma$ satisfying the four-dimensional percentile query

$$\mathcal{Q} := \bigwedge_{i=1}^{4} \mathbb{P}^{\sigma}_{M,s_{\text{init}}}\left[\mathsf{TS}^{T}_{l_i} \leq 0\right] = 1.$$

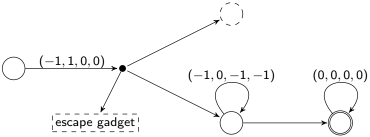
if and only if the machine *does not* halt.

Halting state $\notin$ T: halting $\Rightarrow \mathsf{TS}^{T} = \infty$.

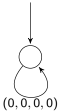
SP with arbitrary weights: undecidability (2/2)



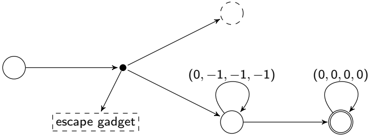
(a) Increment C_1.



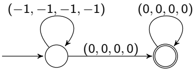
(b) Decrement C_1.



(c) Halting.



(d) Checking a zero test on C_1.



(e) Escape gadget reachable by every action of the MDP.