Controllers in Reactive Synthesis: A Strategic Perspective

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F.R.S.-FNRS & UMONS – Université de Mons, Belgium

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Formal Methods Reading Group, UMONS

The talk in one slide

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But how to define complexity and how to measure it?

↪ **That is our topic of the today.**

The talk in one **more** slide

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Yes, I lied, and I will lie even more. The results I will survey span numerous combinations of

- game models,
- strategy models,
- objectives,
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↪ **I will focus on recent work with marvelous co-authors.**

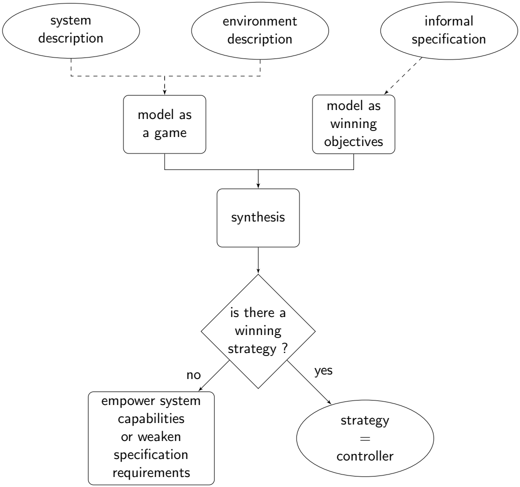
1 Controller synthesis

2 Memory

3 Randomness

4 Beyond Mealy machines

Controller synthesis: a game-theoretic approach



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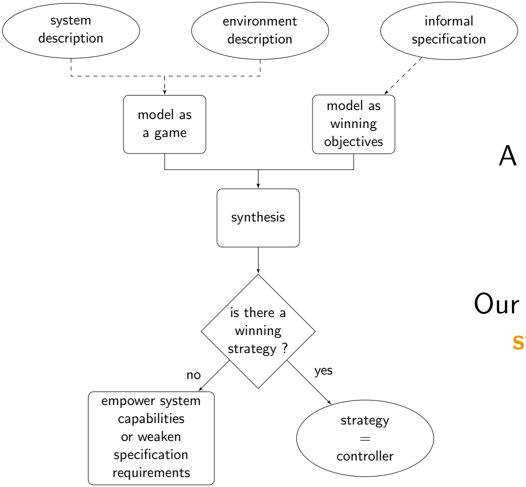
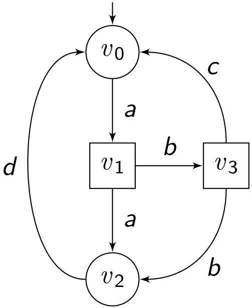


A plethora of models and objectives exist.[1]

Our focus here: how complex **strategies** need to be?

[1] Randour, "Automated Synthesis of Reliable and Efficient Systems Through Game Theory: A Case Study", 2013; Clarke et al., Handbook of Model Checking, 2018; Fijalkow et al., Games on Graphs, 2023.

Two-player games



A two-player turn-based finite **arena** $\mathcal{A} = (V_{\bigcirc}, V_{\square}, E)$ with no deadlock.

Color function $\mathfrak{c} \colon E \to C$.

$\hookrightarrow$ Players move a pebble along the edges creating an infinite **play**.

$\hookrightarrow$ **Behavior of the system = sequence of colors.**

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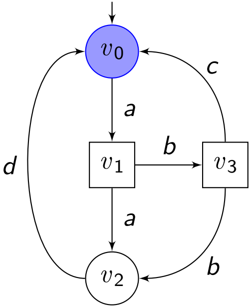

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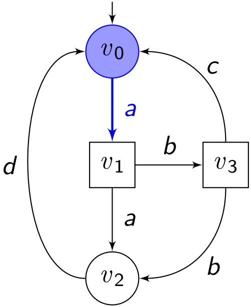

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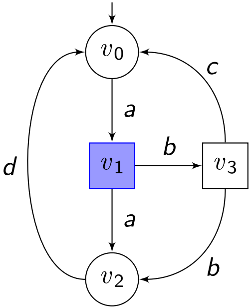

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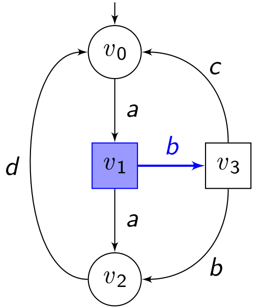

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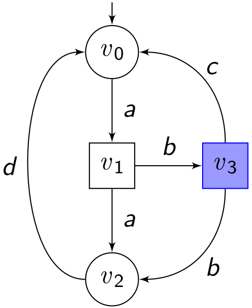

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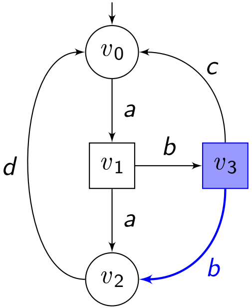

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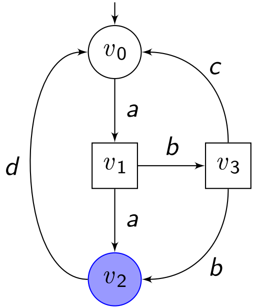

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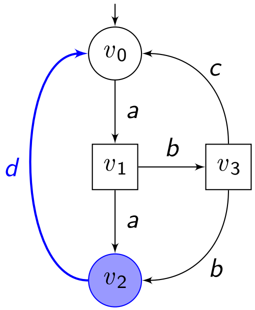

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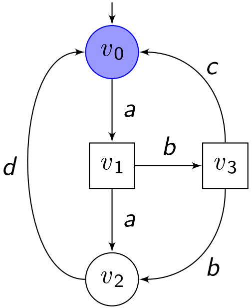

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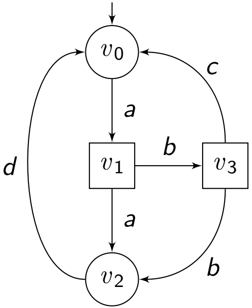

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Sample play: $abbd \ldots \in C^\omega$

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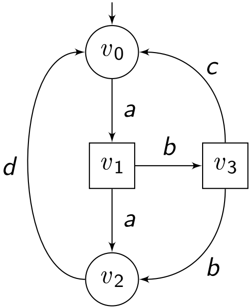

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Usual interpretation

$\mathcal{P}_\bigcirc$ (the system to control) tries to satisfy its **specification** while $\mathcal{P}_\square$ (the environment) tries to prevent it from doing so.

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They are encoded as some kind of *objective* defined using colors.
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1. A **winning condition**: a set of winning plays that $\mathcal{P}_{\bigcirc}$ tries to realize. E.g., $\mathsf{Reach}(t) = \{\pi = c_0 c_1 c_2 \ldots \mid t \in \pi\}$, for $t \in C$ a given color, a *reachability* objective.

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2. A **payoff function** to optimize, assuming $C \subset \mathbb{Q}$. E.g., the *discounted sum* function, defined as $\mathsf{DS}(\pi) = \sum_{i=0}^{\infty} \gamma^i c_i$ for some discount factor $\gamma \in]0, 1[$.

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3. A **preference relation** defines a total preorder over sequences of colors, thus generalizing both previous concepts.

Strategies

Player $\mathcal{P}_\nabla$ chooses outgoing edges following a **strategy**

$$\sigma_\nabla \colon V^* V_\nabla \to V$$

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Optimal strategies (using a preference relation $\sqsubseteq$)

A strategy $\sigma_\bigcirc$ of $\mathcal{P}_\bigcirc$ is optimal if it guarantees (i.e., against an optimal adversary $\mathcal{P}_\Box$) a play at least as good as any other strategy $\sigma'_\bigcirc$ with respect to $\sqsubseteq$.

MDPs & stochastic games

Why?

In many real scenarios, the environment is not fully antagonistic, but exhibits stochastic behaviors.

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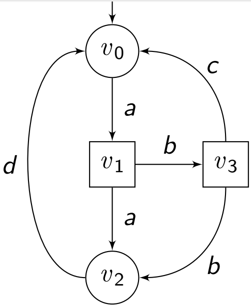

Two-player (deterministic) game.

$$V = V_{\bigcirc} \uplus V_{\square}.$$

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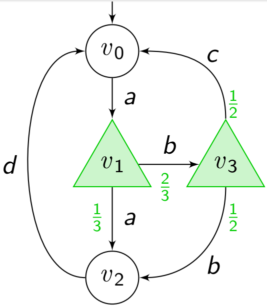

Markov decision process.

$$V = V_{\bigcirc} \uplus V_{\triangle}.$$

Either $\mathcal{P}_{\bigcirc}$ aims to maximize

- $\mathbb{P}^{\sigma_{\bigcirc}}[W]$ for some winning condition W,
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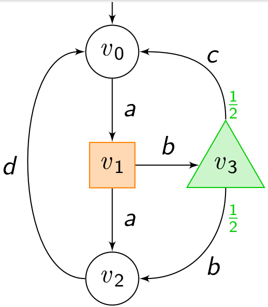

Stochastic game.

$$V = V_{\bigcirc} \uplus V_{\triangle} \uplus V_{\square}.$$

Either $\mathcal{P}_{\bigcirc}$ aims to maximize, against the adversary $\mathcal{P}_{\square}$,

- ▷ $\mathbb{P}^{\sigma_{\bigcirc}, \sigma_{\square}}[W]$ for some winning condition W,
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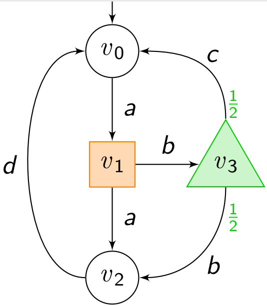


Stochastic game.

$$V = V_{\bigcirc} \uplus V_{\triangle} \uplus V_{\square}.$$

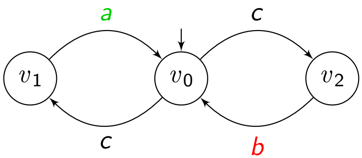
Actions

We often use **actions** instead of stochastic vertices.

Multiple objectives

Combining objectives

Complex objectives arise when combining simple objectives, and usually require more complex strategies to play optimally.



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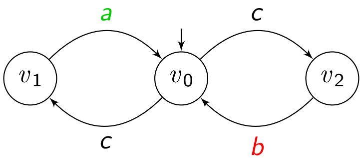
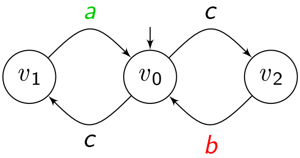
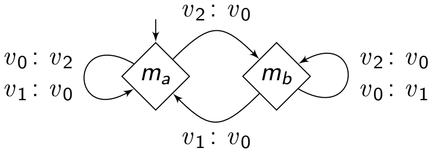


Seeing *a* **and** *b* infinitely often requires memory, but seeing only one does not (Büchi objective).

$\hookrightarrow$ We are often interested in the **Pareto front**, i.e., all payoff vectors not dominated by another.

Classical representation of strategies: Mealy machines





Mealy machine $\mathcal{M} = \{M, m_{\mathsf{init}}, \alpha_{\mathsf{nxt}}, \alpha_{\mathsf{up}}\}$:

- M is the set of *memory states*,
- m_{init} is the *initial state*,
- $\alpha_{\mathsf{nxt}} : M \times V \to V$ is the *next-action function*,
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Finite memory if $|M| < \infty$, memoryless if $|M| = 1$.

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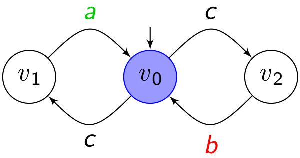

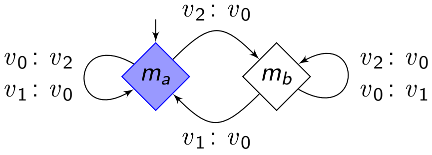

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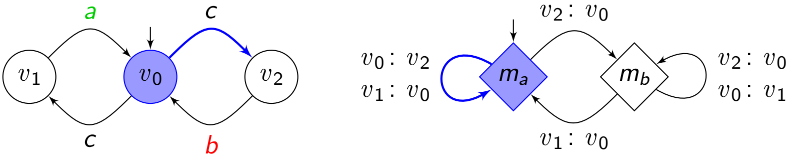

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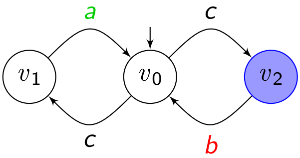

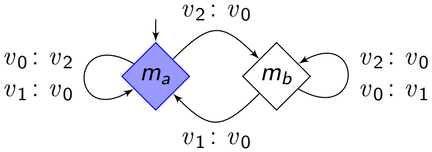

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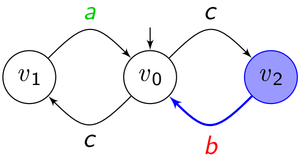

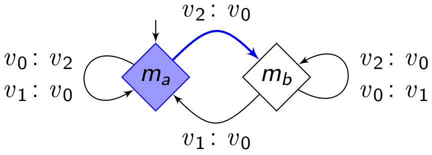

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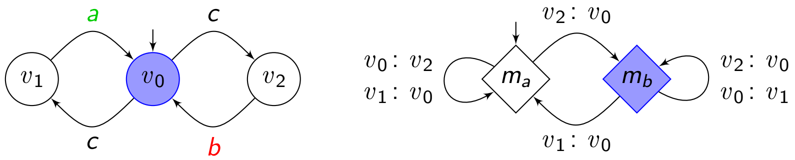

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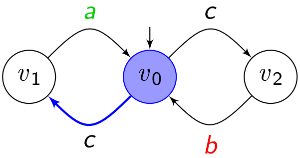

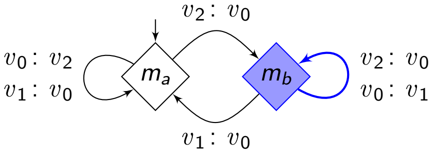

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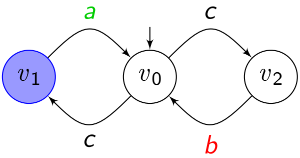

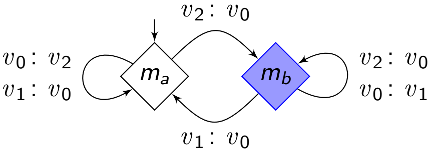

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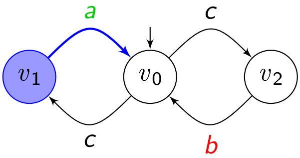

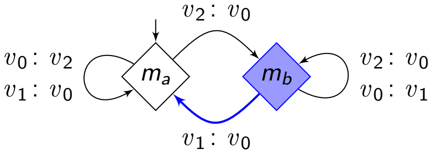

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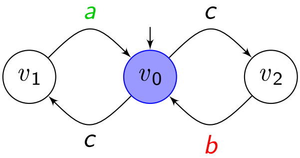

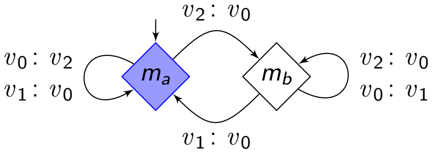

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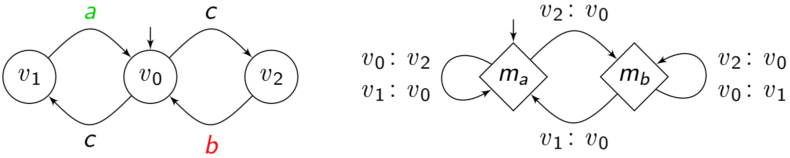

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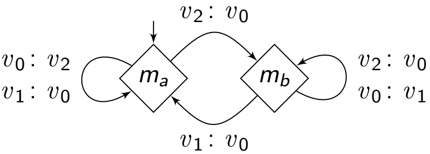

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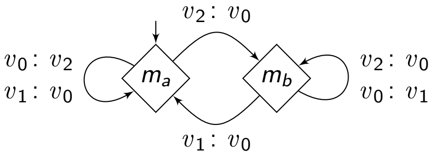

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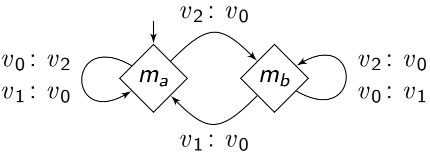

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Many other flavors exist: **chromatic** memory, with or without **ε-transitions**, with different types of **randomness**, etc.

$\hookrightarrow$ **We will discuss some of these.**

Some amazing co-authors

   

Section mostly based on joint work with Patricia Bouyer, Stéphane Le Roux, Youssouf Oualhadj, and Pierre Vandenhove.[2]

[2] Bouyer, Le Roux, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2022; Bouyer, Oualhadj, et al., "Arena-Independent Finite-Memory Determinacy in Stochastic Games", 2023; Bouyer, Randour, and Vandenhove, "Characterizing Omega-Regularity through Finite-Memory Determinacy of Games on Infinite Graphs", 2023.

Memoryless strategies

Functions $\sigma_{\triangledown} \colon V_{\triangledown} \to V$.

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- Equivalently, Mealy machines with one state.
- **Arguably**, the simplest kind of strategies.
- Sufficient to play optimally for most *single* objectives in (stochastic) games: reachability, parity, mean-payoff, discounted sum, etc.

Starting point of our journey: *deterministic* games

Gimbert and Zielonka's characterization[3]

Memoryless strategies suffice (for both players) for a preference relation $\sqsubseteq$ iff it is **monotone** and **selective**.

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Memoryless strategies suffice (for both players) for a preference relation $\sqsubseteq$ iff it is **monotone** and **selective**.

Corollary: one-to-two-player lift

If $\sqsubseteq$ is such that

1. in all $\mathcal{P}_{\bigcirc}$-arenas, $\mathcal{P}_{\bigcirc}$ has optimal memoryless strategies,
2. in all $\mathcal{P}_{\Box}$-arenas, $\mathcal{P}_{\Box}$ has optimal memoryless strategies,

then **both** players have optimal memoryless strategies in all **two-player** arenas.

$\Longrightarrow$ **Extremely useful as analyzing one-player games (i.e., graphs) is much easier.**

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Handling finite-memory strategies (1/3)

Why?

- More complex objectives may require **finite** (multi-Büchi) or **infinite** memory (multi-mean-payoff).

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Unfortunately, it does not hold.

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Let $C \subseteq \mathbb{Z}$ and the winning condition for $\mathcal{P}_{\bigcirc}$ be

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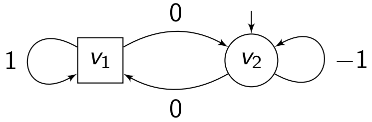

But the two-player one is not!
$\Longrightarrow$ $\mathcal{P}_{\bigcirc}$ **needs infinite memory to win.**

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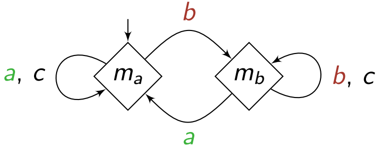

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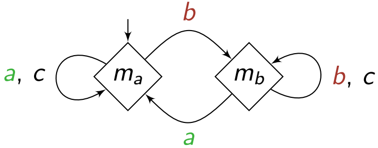

This memory structure **suffices in all arenas**, i.e., it is always possible to find a suitable α_{nxt} to build an optimal Mealy machine.

Handling finite-memory strategies (3/3)

A new frontier

We focus on **arena-independent chromatic** memory structures.

Our characterization[4]

We obtain an equivalent to Gimbert and Zielonka's for finite memory:

1. a characterization through the concepts of $\mathcal{M}$-monotony and $\mathcal{M}$-selectivity,
2. a **one-to-two-player lift**.

[4] Bouyer, Le Roux, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2022.

Extension to stochastic games

We lift[5] this result to **pure arena-independent finite-memory** strategies in **stochastic games**:

1. characterization based on generalizations of $\mathcal{M}$-monotony and $\mathcal{M}$-selectivity,
2. **one-to-two-player lift**, from MDPs to stochastic games.

[5] Bouyer, Oualhadj, et al., "Arena-Independent Finite-Memory Determinacy in Stochastic Games", 2023.

Extension to infinite (deterministic) arenas (1/2)

We consider arenas of arbitrary cardinality and allow infinite branching.

Observation

Memory requirements can be **higher in infinite arenas**: e.g., mean-payoff objectives require infinite memory.

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The case of ω-regular objectives[6]

If a victory condition W is ω-regular, **then** it admits finite-memory optimal strategies in all (infinite) arenas.

[6] Mostowski, "Regular expressions for infinite trees and a standard form of automata", 1985; W. Zielonka, "Infinite games on finitely coloured graphs with applications to automata on infinite trees", 1998.

Extension to infinite (deterministic) arenas (2/2)

The converse[7]

If a **chromatic finite-memory** structure $\mathcal{M}$ suffices for W in all infinite arenas, **then** W is ω-regular.

$\hookrightarrow$ We build a parity automaton for W, based on $\mathcal{M}$ and $\mathcal{S}_W$, the *prefix-classifier* of W (recognizing its Myhill-Nerode classes).

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Corollaries

1. Game-theoretical characterization of ω-regularity.
2. **One-to-two-player lift** for infinite arenas.

[7] Bouyer, Randour, and Vandenhove, "Characterizing Omega-Regularity through Finite-Memory Determinacy of Games on Infinite Graphs", 2023.

Other criteria and characterizations

There is a plethora of results related to memory (models vary). Non-exhaustive list:

- characterizations through universal graphs,[8]
- tight memory bounds for sub-classes of objectives,[9]
- criteria for half-positionality,[10]
- one-to-multi-objective lift,[11]
- two-to-multi-player lift.[12]

↪ Find more about **chromatic memory** in our survey.[13]

[8] Casares and Ohlmann, "Characterising Memory in Infinite Games", 2023.

[9] Bouyer, Casares, et al., "Half-Positional Objectives Recognized by Deterministic Büchi Automata", 2024; Bouyer, Fijalkow, et al., "How to Play Optimally for Regular Objectives?", 2023; Casares and Ohlmann, "Positional ω-regular languages", 2024.

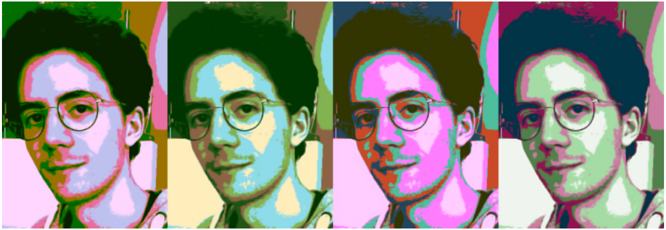
[10] Aminof and Rubin, "First-cycle games", 2017.

[11] Le Roux, Pauly, and Randour, "Extending Finite-Memory Determinacy by Boolean Combination of Winning Conditions", 2018.

[12] Le Roux and Pauly, "Extending Finite Memory Determinacy to Multiplayer Games", 2016.

[13] Bouyer, Randour, and Vandenhove, "The True Colors of Memory: A Tour of Chromatic-Memory Strategies in Zero-Sum Games on Graphs (Invited Talk)", 2022.

The amazing Mr. Main



Section mostly based on (ongoing) joint work with James C. A. Main.[14]

[14] Main and Randour, "Different Strokes in Randomised Strategies: Revisiting Kuhn's Theorem Under Finite-Memory Assumptions", 2024.

Introducing randomness in strategies (1/2)

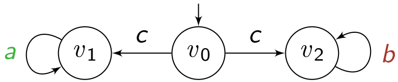
A **pure** strategy is a function $\sigma_{\nabla} \colon V^* V_{\nabla} \to V$.

Introducing randomness in strategies (1/2)

A **pure** strategy is a function $\sigma_{\bigtriangledown} \colon V^* V_{\bigtriangledown} \to V$.

We may need **randomness** to deal with, e.g.,

- ▷ multiple objectives,
- ▷ concurrent games,
- ▷ imperfect information.



Objective: $\mathbb{P}^{\sigma_{\bigcirc}}[\text{Reach}(a)] \geq \frac{1}{2} \land \mathbb{P}^{\sigma_{\bigcirc}}[\text{Reach}(b)] \geq \frac{1}{2}$

$\hookrightarrow$ **Achievable by tossing a coin in v_0.**

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$\mathcal{D}(\sigma_{\nabla} \colon V^* V_{\nabla} \to V)$

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Kuhn's theorem[15]

All three classes are equivalent in games of *perfect recall*.

$\hookrightarrow$ **Requires access to infinite memory and infinite support for distributions.**

[15] Aumann, "Mixed and Behavior Strategies in Infinite Extensive Games", 1964.

What about finite-memory strategies?

Mealy machine $\mathcal{M} = \{M, m_{\mathsf{init}}, \alpha_{\mathsf{nxt}}, \alpha_{\mathsf{up}}\}$:

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Stochastic Mealy machine $\mathcal{M} = \{M, \mu_{\mathsf{init}}, \alpha_{\mathsf{nxt}}, \alpha_{\mathsf{up}}\}$:

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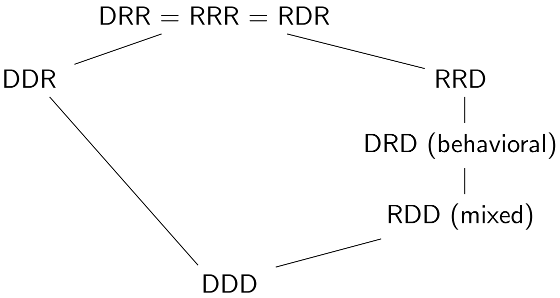
$\Longrightarrow$ **Three ways to add randomness: initialization, outputs, and updates.**

Taxonomy[16] (1/2)



Classes XYZ where X, Y, Z $\in \{D, R\}$ where D stands for deterministic and R for random, and

- X characterizes the initialization,
- Y characterizes the next-action function,
- Z characterizes the update function.

[16] Main and Randour, "Different Strokes in Randomised Strategies: Revisiting Kuhn's Theorem Under Finite-Memory Assumptions", 2024.

Taxonomy (2/2)

This taxonomy holds from one-player deterministic games (no collapse) up to concurrent partial-information multi-player games (equivalences hold).

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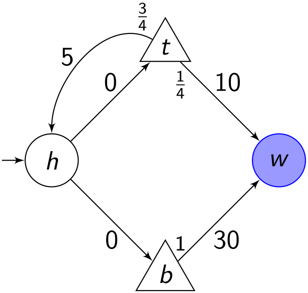
$\hookrightarrow$ **Collapses may arise for restricted classes of objectives (WiP).**

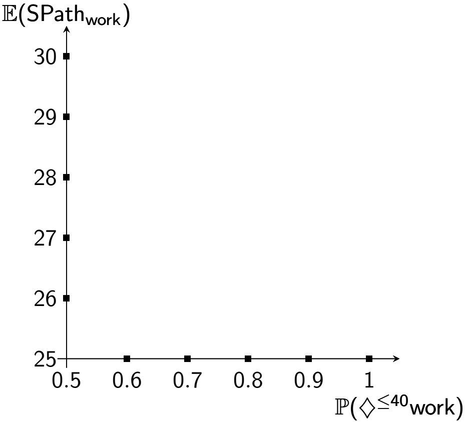
Multi-objectives MDPs (1/2)

We consider **two goals**:

- reaching work under 40 minutes with **high probability**;
- minimizing the **expectancy** of the time to reach work.



From *home*, take the *train* or *bike* to reach *work*.



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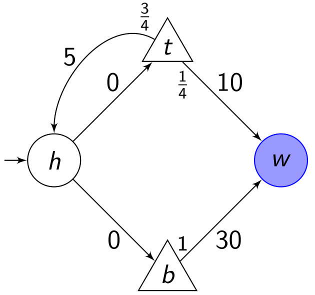

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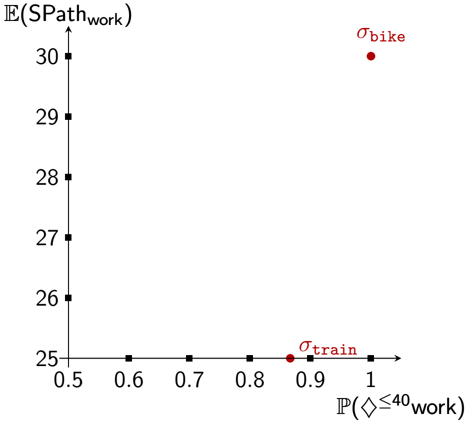

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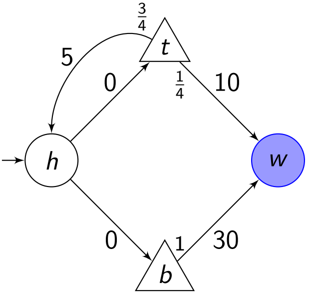

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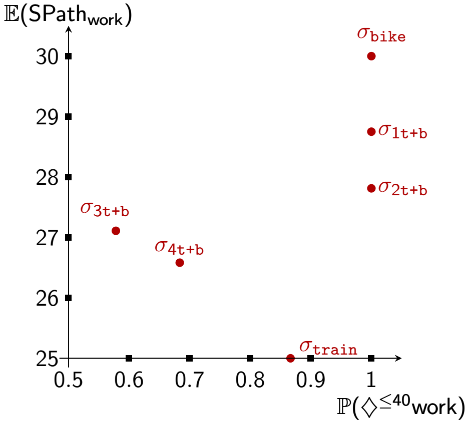

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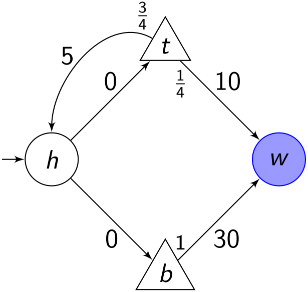

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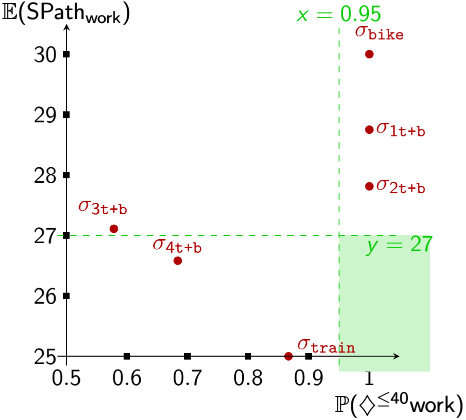

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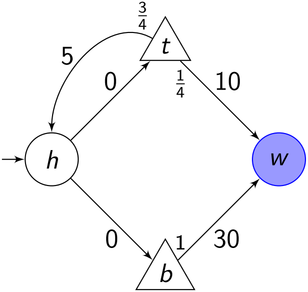

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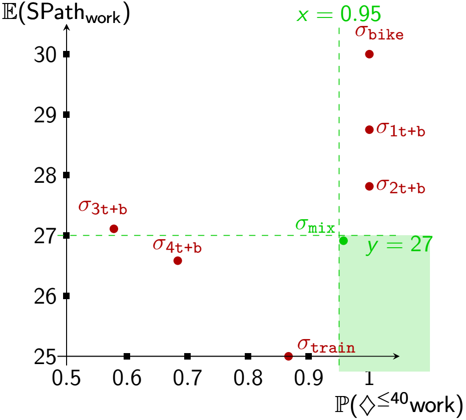

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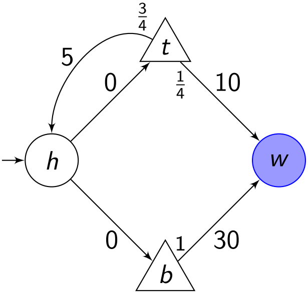

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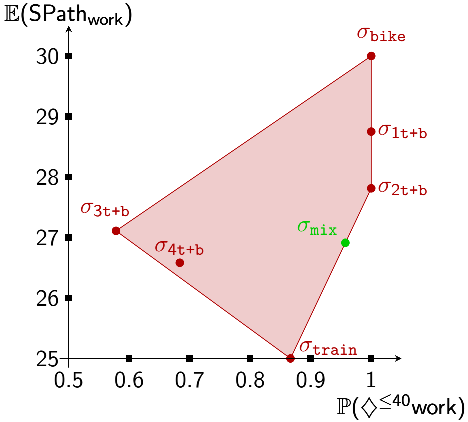

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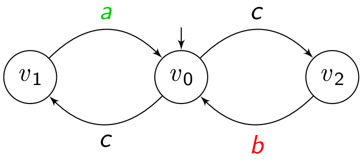
$\Longrightarrow$ **RDD-randomization is sufficient in most multi-objective MDPs.**

Trading memory for randomness

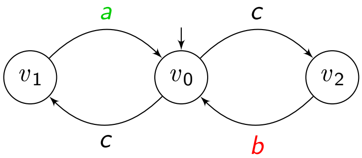
Recall this generalized Büchi game asking to see a and b infinitely often:



We need (a two-state) memory to win it with *pure* strategies.

Trading memory for randomness

Recall this generalized Büchi game asking to see *a* and *b* infinitely often:



We need (a two-state) memory to win it with *pure* strategies.

But a (behavioral) randomized memoryless strategy suffices to win with probability one: playing v_1 and v_2 with non-zero probability ensures it.

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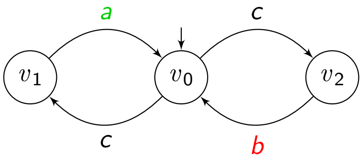

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$\hookrightarrow$ **Memory can be traded for randomness for some classes of games/objectives.**[17]

[17] Chatterjee, de Alfaro, and Henzinger, "Trading Memory for Randomness", 2004; Chatterjee, Randour, and Raskin, "Strategy synthesis for multi-dimensional quantitative objectives", 2014.

An incomplete story

Leitmotiv

Simpler strategies are better (for controller synthesis).

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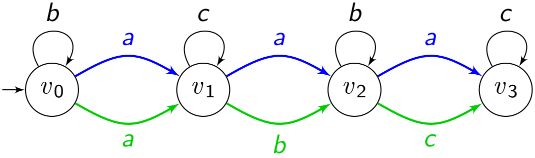
$\hookrightarrow$ **Let us question that.**

All memoryless strategies are not created equal

We want to reach v_3.



Intuitively, the blue strategy seems simpler than the green one.

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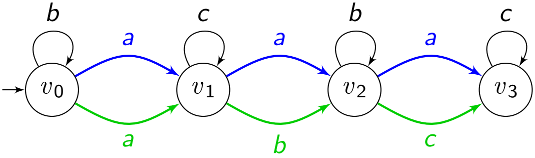

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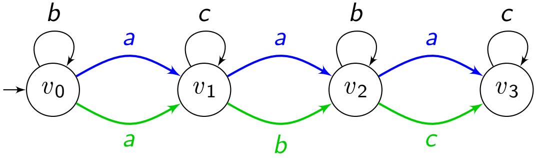

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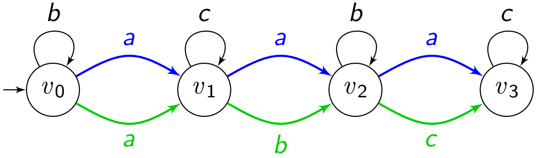

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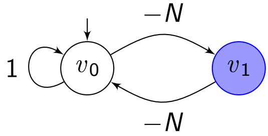
↪ **Memoryless strategies can already be too large to represent in practice!**

Memory is not always an issue

Multi-objectives games involving payoffs often require **exponential memory**. E.g., energy-Büchi objective with $N \in \mathbb{N}$.



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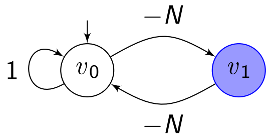

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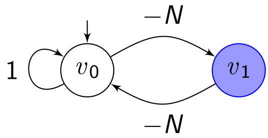

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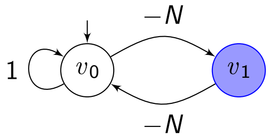

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Hot take

We should explore novel notions of **simplicity**, and consider *alternative representations* of strategies/controllers.

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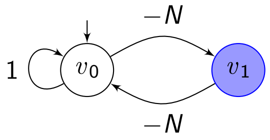

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Hot take

We should explore novel notions of **simplicity**, and consider *alternative representations* of strategies/controllers.

↪ **We quickly survey a few ones in the next slides.**

Structurally-enriched Mealy machines

Idea:

- ▷ Augment Mealy machines with **data structures**: e.g., counters.[18]
- ▷ Avoid "flattening" structural information about the strategy: better understandability and closer to actual controllers.
- ▷ Link with James's talk about interval strategies in OC-MDPs.

$\Longrightarrow$ **Changes our way of thinking which strategies are complex or not.**

[18] Blahoudek et al., "Qualitative Controller Synthesis for Consumption Markov Decision Processes", 2020.

Decision trees

- Structured state-space (e.g., $\subset \mathbb{Z}^n$) and action-space.
- Learn a (possibly approximative) decision tree from a given **memoryless** strategy.
- More understandable and compact than huge action tables.
- More complex tests may reduce size but hinder readability.

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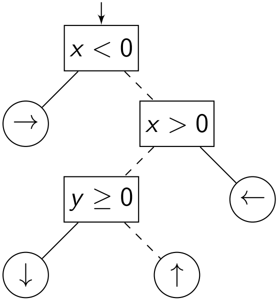
Toy example: trying to reach the center $(0,0)$ of a 2D-grid.



instead of

x	y	action
0	1	$\downarrow$
0	2	$\downarrow$
...	...	$\downarrow$
-1	0	$\rightarrow$
-1	1	$\rightarrow$
...	...	...

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Works well in practice. . . [19]

. . . starting from a given memoryless strategy.

[19] Brazdil, Chatterjee, Chmelik, et al., "Counterexample Explanation by Learning Small Strategies in Markov Decision Processes", 2015; Brazdil, Chatterjee, Kretinsky, et al., "Strategy Representation by Decision Trees in Reactive Synthesis", 2018.

Other alternatives

- **Programmatic representations.**
  - ▷ Closer to realistic code, understandable.
  - ▷ Strongly linked to the input format of the problem (e.g., PRISM code[20]), hard to generalize.
- **Models inspired by Turing machines.**
  - ▷ Powerful but hard to work with.
  - ▷ Tentative notion of decision speed.[21]
- **Neural networks.**
  - ▷ Prevalent in RL.
  - ▷ Hard to understand and verify.
  - ▷ Can be coupled with finite-state-machine abstractions.[22]

[22] Shabadi, Fijalkow, and Matricon, "Theoretical foundations for programmatic reinforcement learning", 2024.

[22] Gelderie, "Strategy machines: representation and complexity of strategies in infinite games", 2014.

[22] Carr, Jansen, and Topcu, "Verifiable RNN-Based Policies for POMDPs Under Temporal Logic Constraints", 2020.

Wrap-up

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Mealy machines are a powerful tool from a theoretical standpoint.

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Strategy complexity $\neq$ representation complexity.

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$\hookrightarrow$ **Many questions are still open!**

Strategy complexity $\neq$ representation complexity.

Take-home message

We need a **proper theory of complexity**, and a **toolbox of different representations**.

$\hookrightarrow$ **Ongoing project ControlleRS.**

Thank you! Any question?