# Simplicity Lies in the Eye of the Beholder
—
# A Strategic Perspective on Controllers in Reactive Synthesis

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F.R.S.-FNRS & UMONS – Université de Mons, Belgium

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*Reachability Problems 2025*

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**But how to define complexity and how to measure it?**

↪ **That is our topic of the today.**

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↪ **I will focus on recent work with marvelous co-authors.**

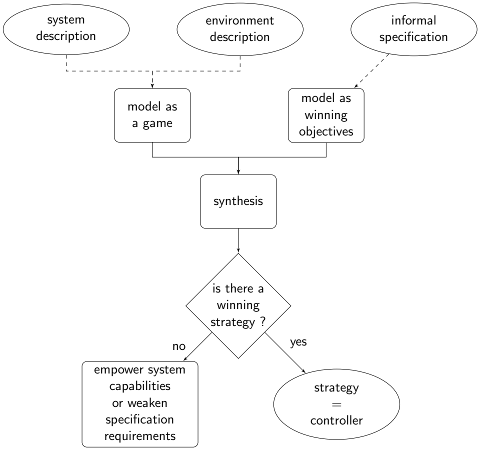
1 Controller synthesis

2 Memory

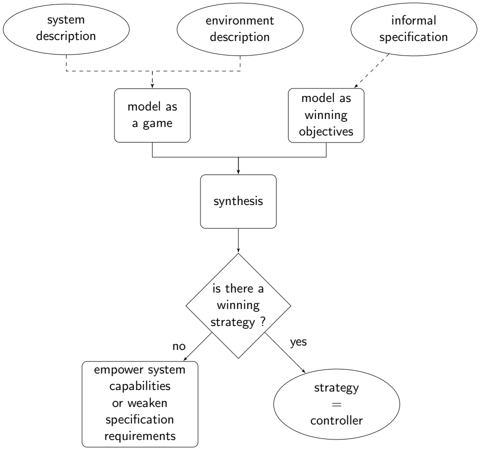
3 Randomness

4 Beyond Mealy machines

# Controller synthesis: a game-theoretic approach



# Controller synthesis: a game-theoretic approach

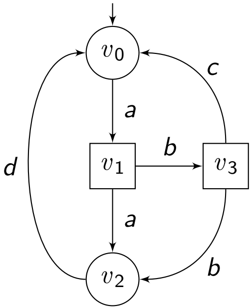



A plethora of models and objectives exist.[1]

Our focus here: how complex **strategies** need to be?

[1] Randour, "Automated Synthesis of Reliable and Efficient Systems Through Game Theory: A Case Study", 2013; Bloem, Chatterjee, and Jobstmann, "Graph Games and Reactive Synthesis", 2018; Fijalkow et al., Games on Graphs: From Logic and Automata to Algorithms, 2025.

# Two-player games




A two-player turn-based finite **arena** $\mathcal{A} = (V_{\bigcirc}, V_{\square}, E)$ with no deadlock.

**Color** function $\mathfrak{c}\colon E \to C$.

$\hookrightarrow$ Players move a pebble along the edges creating an infinite **play**.

$\hookrightarrow$ **Behavior of the system = sequence of colors.**

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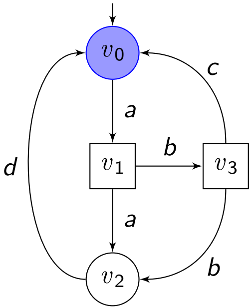


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Sample play:

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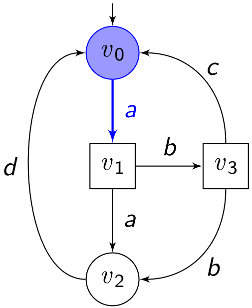


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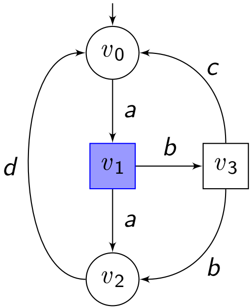


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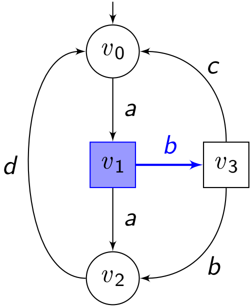


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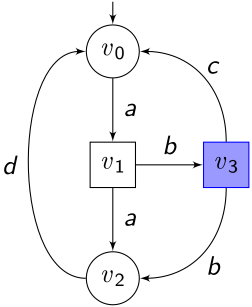


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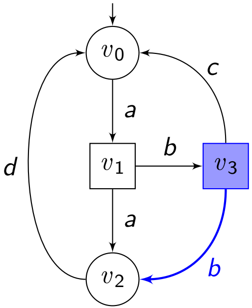


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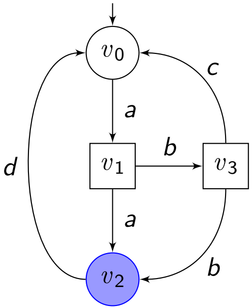


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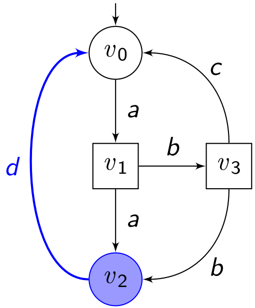


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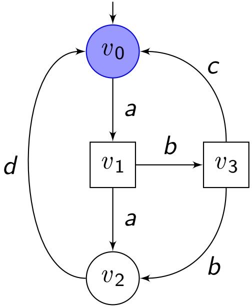


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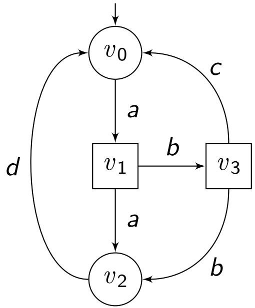


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Sample play: $abbd\ldots \in C^{\omega}$

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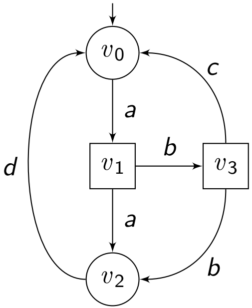


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## Usual interpretation

$\mathcal{P}_\bigcirc$ (the system to control) tries to satisfy its **specification** while $\mathcal{P}_\Box$ (the environment) tries to prevent it from doing so.

# Specifications

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1. A **winning condition**: a set of winning plays that $\mathcal{P}_{\bigcirc}$ tries to realize. E.g., $\mathsf{Reach}(t) = \{\pi = c_0 c_1 c_2 \ldots \mid t \in \pi\}$, for $t \in C$ a given color, a *reachability* objective.

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2. A **payoff function** to optimize, assuming $C \subset \mathbb{Q}$. E.g., the *discounted sum* function, defined as $\text{DS}(\pi) = \sum_{i=0}^{\infty} \gamma^i c_i$ for some discount factor $\gamma \in (0, 1)$.

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3. A **preference relation** defines a total preorder over sequences of colors, thus generalizing both previous concepts.

# Strategies

Player $\mathcal{P}_\nabla$ chooses outgoing edges following a **strategy**

$$\sigma_\nabla \colon (V\,E)^* V_\nabla \to E$$

consistent with the underlying graph.

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**Optimal strategies (using a preference relation $\sqsubseteq$)**

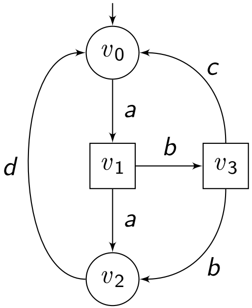
A strategy $\sigma_\bigcirc$ of $\mathcal{P}_\bigcirc$ is optimal if its worst-case outcome (i.e., considering all strategies of $\mathcal{P}_\Box$) is at least as good, with respect to $\sqsubseteq$, as that of any other strategy $\sigma'_\bigcirc$.

# MDPs & stochastic games

**Why?**

In many real-world scenarios, the environment is not fully antagonistic, but exhibits stochastic behaviors.

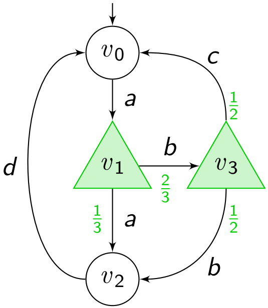
# MDPs & stochastic games




Two-player (deterministic) game.

$$V = V_{\bigcirc} \uplus V_{\square}.$$

# MDPs & stochastic games




**Markov decision process.**

$$V = V_{\bigcirc} \uplus V_{\triangle}.$$

Either $\mathcal{P}_{\bigcirc}$ aims to maximize

- $\mathbb{P}^{\sigma_{\bigcirc}}[W]$ for some winning condition $W$,
- or $\mathbb{E}^{\sigma_{\bigcirc}}[f]$ for some payoff function $f$.

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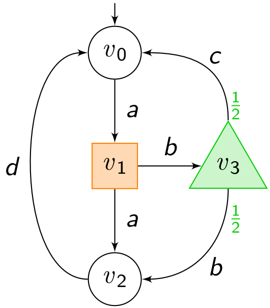


**Stochastic game.**

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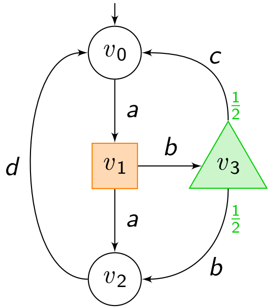


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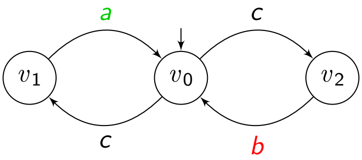
## Actions

We often use **actions** instead of stochastic vertices.

# Multiple objectives

## Combining objectives

Complex objectives arise when combining simple objectives, and usually require more complex strategies to play optimally.

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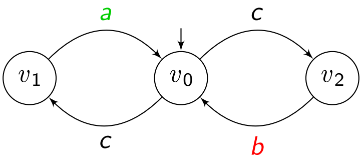
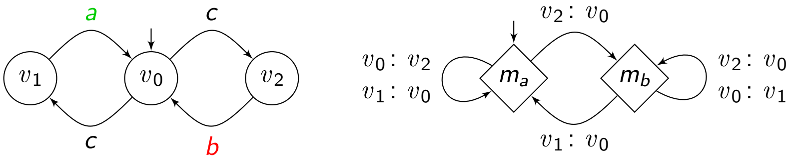



Seeing $a$ **and** $b$ infinitely often requires memory, but seeing only one does not (Büchi objective).

↪ We are often interested in the **Pareto frontier**, i.e., all payoff vectors not dominated by another.

# Classical representation of strategies: Mealy machines




**Mealy machine** $\mathcal{M} = (M, m_{\text{init}}, \alpha_{\text{nxt}}, \alpha_{\text{up}})$:

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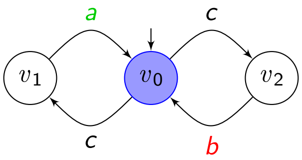


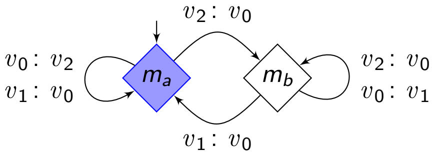


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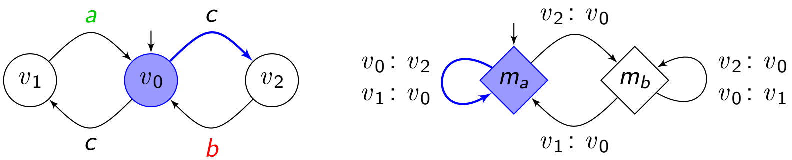


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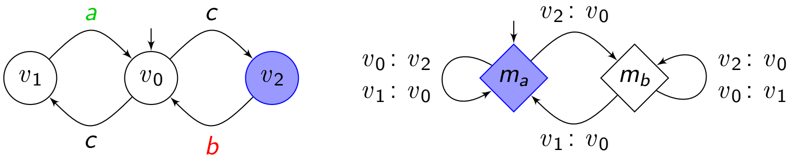


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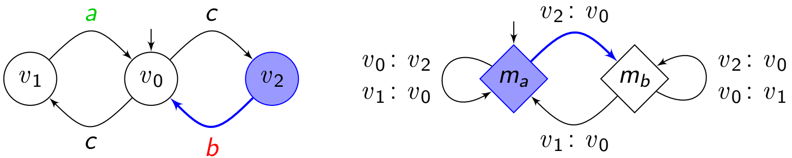


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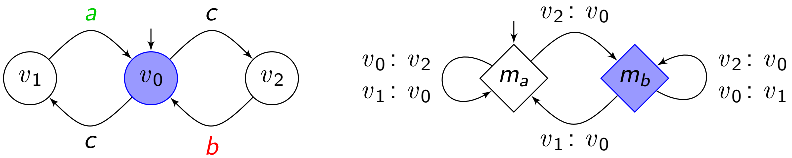


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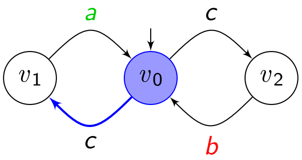


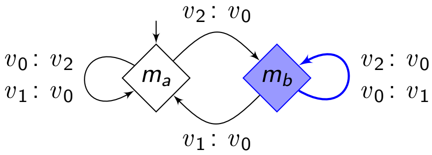


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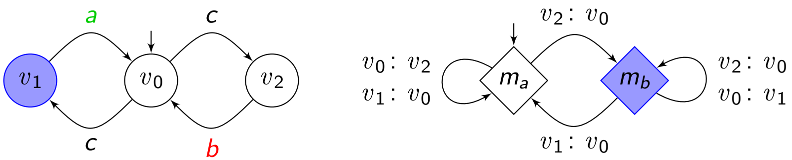


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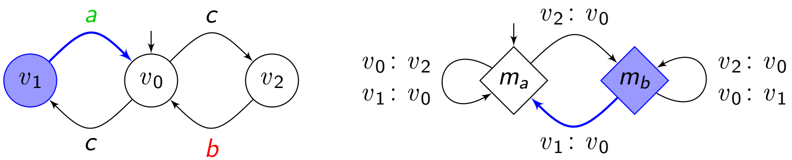


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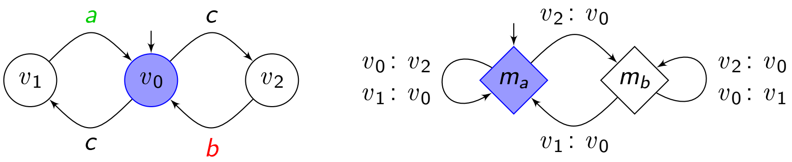


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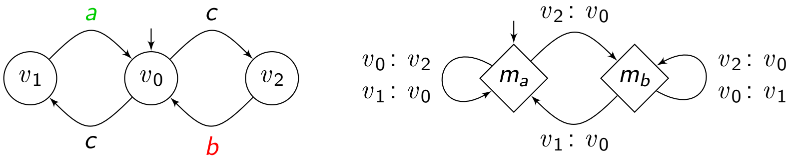


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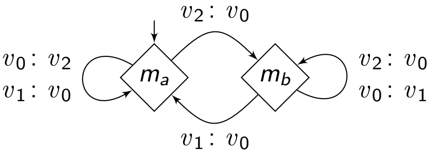


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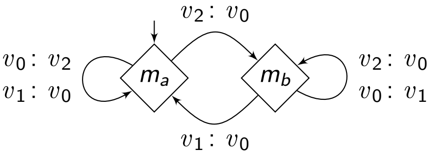


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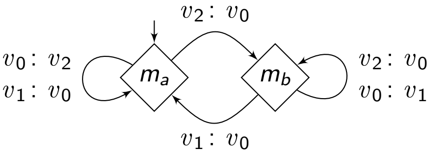


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Many other flavors exist: **chromatic** memory, with or without **$\varepsilon$-transitions**, with different types of **randomness**, etc.

$\hookrightarrow$ **We will discuss some of these.**

# Some amazing co-authors




Section mostly based on joint work with Patricia Bouyer, Stéphane Le Roux, Youssouf Oualhadj, and Pierre Vandenhove.[2]

[2] Bouyer, Le Roux, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2022; Bouyer, Oualhadj, et al., "Arena-Independent Finite-Memory Determinacy in Stochastic Games", 2023; Bouyer, Randour, and Vandenhove, "Characterizing Omega-Regularity through Finite-Memory Determinacy of Games on Infinite Graphs", 2023.

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Functions $\sigma_{\triangledown} \colon V_{\triangledown} \to E$.

- Equivalently, Mealy machines with one state.
- **Arguably**, the simplest kind of strategies.
- Sufficient to play optimally for most *single* objectives in (stochastic) games: reachability, parity, mean-payoff, discounted sum, etc.

# Starting point of our journey: *deterministic* games

**Gimbert and Zielonka's characterization[3]**

Memoryless strategies suffice (for both players) for a preference relation $\sqsubseteq$ iff $\sqsubseteq$ and $\sqsubseteq^{-1}$ are **monotone** and **selective**.

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## Corollary: one-to-two-player lift

If $\sqsubseteq$ is such that

1. in all $\mathcal{P}_{\bigcirc}$-arenas, $\mathcal{P}_{\bigcirc}$ has optimal memoryless strategies,
2. in all $\mathcal{P}_{\Box}$-arenas, $\mathcal{P}_{\Box}$ has optimal memoryless strategies,

then **both** players have optimal memoryless strategies in all **two-player** arenas.

$\Longrightarrow$ **Extremely useful as analyzing one-player games (i.e., graphs) is much easier.**

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# Handling finite-memory strategies (1/3)

Why?

- More complex objectives may require **finite** (multi-Büchi) or **infinite** memory (multi-mean-payoff).

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**Unfortunately, it does not hold.**

# Handling finite-memory strategies (2/3)

Let $C \subseteq \mathbb{Z}$ and the winning condition for $\mathcal{P}_{\bigcirc}$ be

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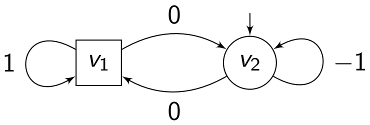


**But the two-player one is not!**
**$\Longrightarrow$ $\mathcal{P}_{\bigcirc}$ needs infinite memory to win.**

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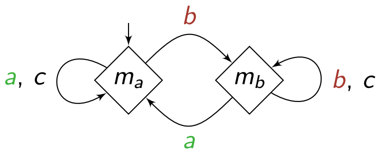

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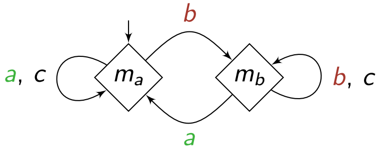


This memory structure **suffices in all arenas**, i.e., it is always possible to find a suitable $\alpha_{\mathsf{nxt}}$ to build an optimal Mealy machine.

# Handling finite-memory strategies (3/3)

## A new frontier

We focus on **arena-independent chromatic** memory structures.

## Our characterization[4]

We obtain an equivalent to Gimbert and Zielonka's for finite memory:

1. a characterization through the concepts of $\mathcal{M}$-monotony and $\mathcal{M}$-selectivity,
2. a **one-to-two-player lift**.

[4] Bouyer, Le Roux, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2022.

# Extension to stochastic games

We lift[5] this result to **pure arena-independent finite-memory** strategies in **stochastic games**:

1. characterization based on generalizations of $\mathcal{M}$-monotony and $\mathcal{M}$-selectivity,
2. **one-to-two-player lift**, from MDPs to stochastic games.

[5] Bouyer, Oualhadj, et al., "Arena-Independent Finite-Memory Determinacy in Stochastic Games", 2023.

# Extension to infinite (deterministic) arenas (1/2)

We consider arenas of arbitrary cardinality and allow infinite branching.

**Observation**

Memory requirements can be **higher in infinite arenas**: e.g., mean-payoff objectives require infinite memory.

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**The case of $\omega$-regular objectives[6]**

If a winning condition $W$ is $\omega$-regular, **then** it admits finite-memory optimal strategies in all (infinite) arenas.

[6] Mostowski, "Regular expressions for infinite trees and a standard form of automata", 1985; Zielonka, "Infinite games on finitely coloured graphs with applications to automata on infinite trees", 1998.

# Extension to infinite (deterministic) arenas (2/2)

**The converse**[7]

If a **chromatic finite-memory** structure $\mathcal{M}$ suffices for $W$ in all infinite arenas, **then** $W$ is $\omega$-regular.

$\hookrightarrow$ We build a parity automaton for $W$, based on $\mathcal{M}$ and $\mathcal{S}_W$, the *prefix-classifier* of $W$ (recognizing its Myhill-Nerode classes).

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**Corollaries**

1. Game-theoretical characterization of $\omega$-regularity.
2. **One-to-two-player lift** for infinite arenas.

[7] Bouyer, Randour, and Vandenhove, "Characterizing Omega-Regularity through Finite-Memory Determinacy of Games on Infinite Graphs", 2023.

# Other criteria and characterizations

There is a plethora of results related to memory (models vary). Non-exhaustive list:

- characterizations through universal graphs,[8]
- tight memory bounds for sub-classes of objectives,[9]
- criteria for half-positionality,[10]
- one-to-multi-objective lift,[11]
- two-to-multi-player lift.[12]

$\hookrightarrow$ Find more about **chromatic memory** in our survey.[13]

---

[8] Casares and Ohlmann, "Characterising memory in infinite games", 2025.

[9] Bouyer, Casares, et al., "Half-Positional Objectives Recognized by Deterministic Büchi Automata", 2024; Bouyer, Fijalkow, et al., "How to Play Optimally for Regular Objectives?", 2023; Casares and Ohlmann, "Positional $\omega$-regular languages", 2024.

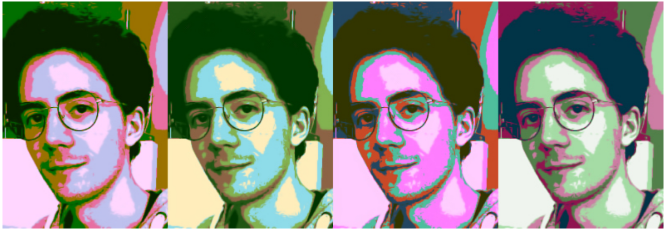
[10] Kopczyński, "Half-positional Determinacy of Infinite Games", 2008.

[11] Le Roux, Pauly, and Randour, "Extending Finite-Memory Determinacy by Boolean Combination of Winning Conditions", 2018.

[12] Le Roux and Pauly, "Extending Finite Memory Determinacy to Multiplayer Games", 2016.

[13] Bouyer, Randour, and Vandenhove, "The True Colors of Memory: A Tour of Chromatic-Memory Strategies in Zero-Sum Games on Graphs", 2022.

# The amazing Mr. Main



Section mostly based on joint work with
James C. A. Main.[14]

[14] Main and Randour, "Different Strokes in Randomised Strategies: Revisiting Kuhn's Theorem Under Finite-Memory Assumptions", 2024; Main and Randour, "Mixing Any Cocktail with Limited Ingredients: On the Structure of Payoff Sets in Multi-Objective MDPs and its Impact on Randomised Strategies", 2025.

# Introducing randomness in strategies (1/2)

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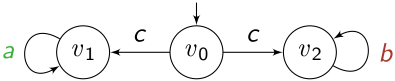
A **pure** strategy is a function $\sigma_{\triangledown}\colon (V\,E)^* V_{\triangledown} \to E$.

We may need **randomness** to deal with, e.g.,

- ▷ multiple objectives,
- ▷ concurrent games,
- ▷ imperfect information.




Objective: $\mathbb{P}^{\sigma_\bigcirc}[\text{Reach}(a)] \geq \frac{1}{2} \wedge \mathbb{P}^{\sigma_\bigcirc}[\text{Reach}(b)] \geq \frac{1}{2}$

$\hookrightarrow$ **Achievable by tossing a coin in $v_0$.**

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**Kuhn's theorem**[15]

All three classes are equivalent in games of *perfect recall*.

$\hookrightarrow$ **Requires access to infinite memory and infinite support for distributions.**

[15] Aumann, "Mixed and Behavior Strategies in Infinite Extensive Games", 1964; Bertrand, Genest, and Gimbert, "Qualitative Determinacy and Decidability of Stochastic Games with Signals", 2017.

# What about finite-memory strategies?

**Mealy machine** $\mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{nxt}}, \alpha_{\mathsf{up}})$:

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**Stochastic Mealy machine** $\mathcal{M} = (M, \mu_{\mathsf{init}}, \alpha_{\mathsf{nxt}}, \alpha_{\mathsf{up}})$:

- $M$ is the set of memory states,
- $\mu_{\mathsf{init}} \in \mathcal{D}(M)$ is the initial distribution,
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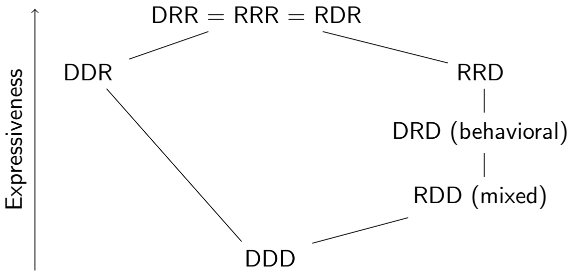
$\Longrightarrow$ **Three ways to add randomness: initialization, outputs, and updates.**

# Taxonomy[16] (1/2)




Classes XYZ with X, Y, Z $\in \{D, R\}$, where D stands for deterministic and R for random, and

- X characterizes the initialization,
- Y characterizes the next-action function,
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[16] Main and Randour, "Different Strokes in Randomised Strategies: Revisiting Kuhn's Theorem Under Finite-Memory Assumptions", 2024.

# Taxonomy (2/2)

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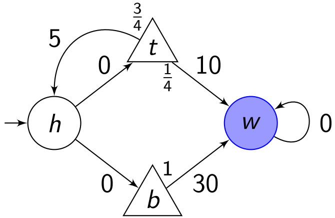
$\hookrightarrow$ **Collapses may arise for restricted classes of objectives (WiP).**

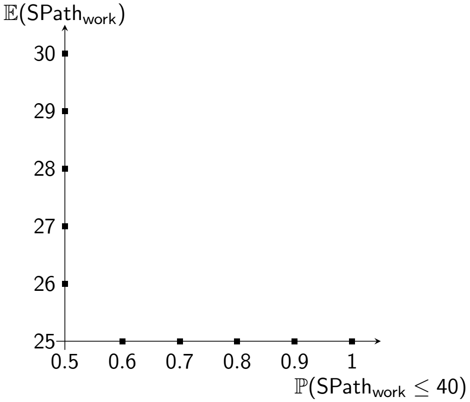
# Multi-objectives MDPs (1/2)

We consider **two goals**:

- reaching work under 40 minutes with **high probability**;
- minimizing the **expectancy** of the time to reach work.




From *home*, take the *train* or *bike* to reach *work*.



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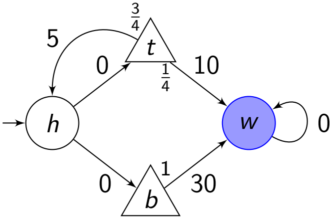


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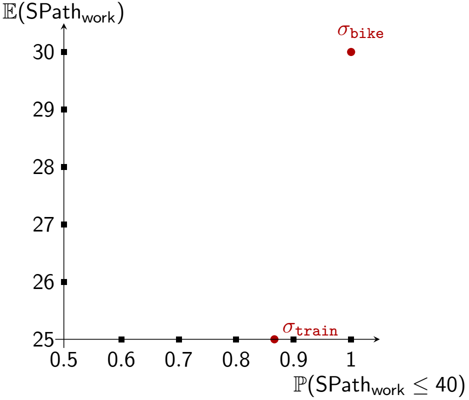

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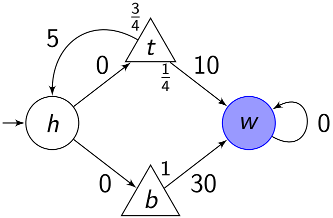


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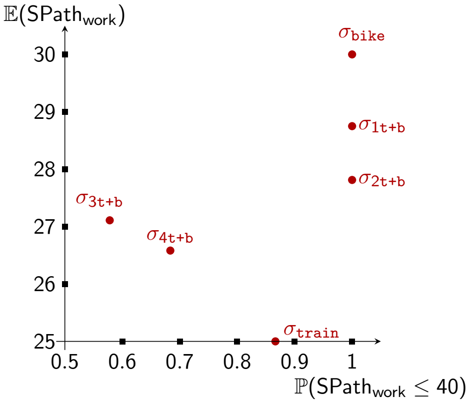

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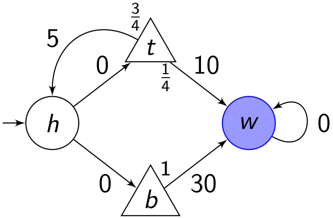


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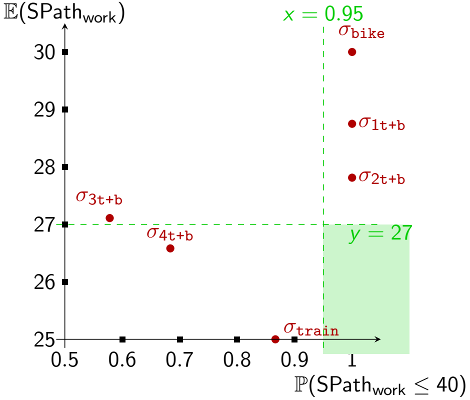

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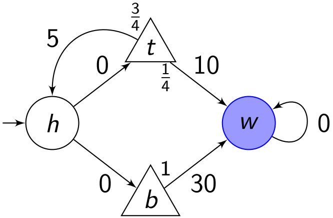


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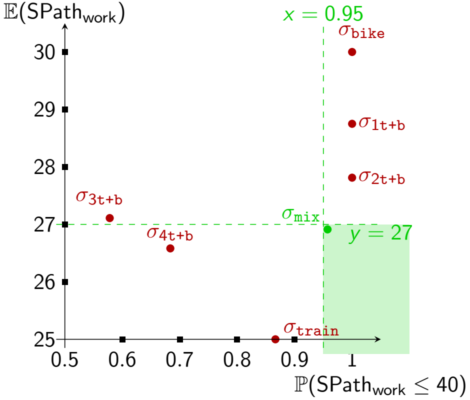

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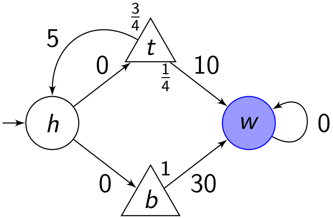


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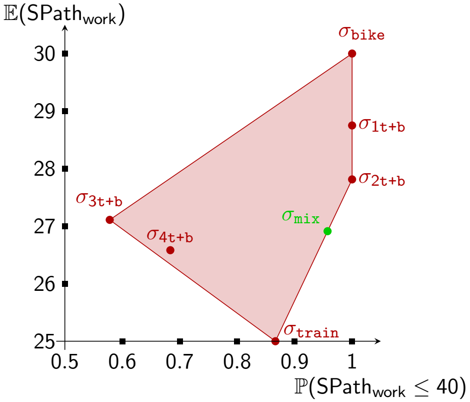

# Multi-objectives MDPs (2/2)

We are interested in the structure of this **payoff set**.

## Our result[17]

For *good* payoff functions ($\sim$ expectancy is well-defined),

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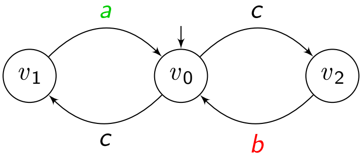
$\Longrightarrow$ **RDD-randomization is sufficient in most multi-objective MDPs.**

[17] Main and Randour, "Mixing Any Cocktail with Limited Ingredients: On the Structure of Payoff Sets in Multi-Objective MDPs and its Impact on Randomised Strategies", 2025.

# Trading memory for randomness

Recall this generalized Büchi game asking to see $a$ and $b$ infinitely often:




We need (a two-state) memory to win it with *pure* strategies.

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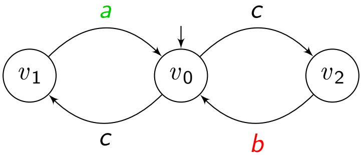


We need (a two-state) memory to win it with *pure* strategies.

**But a (behavioral) randomized memoryless strategy suffices to win with probability one**: playing $v_1$ and $v_2$ with non-zero probability ensures it.

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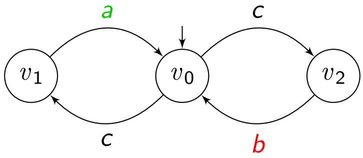


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**But a (behavioral) randomized memoryless strategy suffices to win with probability one**: playing $v_1$ and $v_2$ with non-zero probability ensures it.

$\hookrightarrow$ **Memory can be traded for randomness for some classes of games/objectives.**[18]

[18] Chatterjee, de Alfaro, and Henzinger, "Trading Memory for Randomness", 2004; Chatterjee, Randour, and Raskin, "Strategy synthesis for multi-dimensional quantitative objectives", 2014.

# An incomplete story

**Leitmotiv**

Simpler strategies are better (for controller synthesis).

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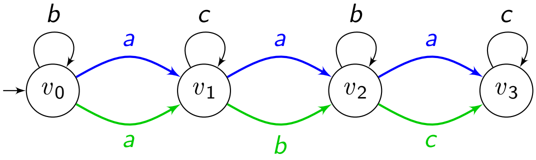
**Usual answer: small memory, no randomness.**

$\hookrightarrow$ **Let us question that.**

# Not all memoryless strategies are created equal

We want to reach $v_3$.

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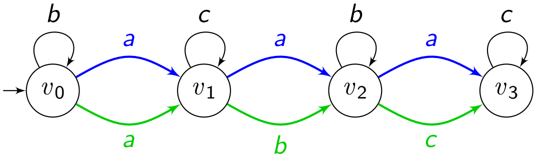


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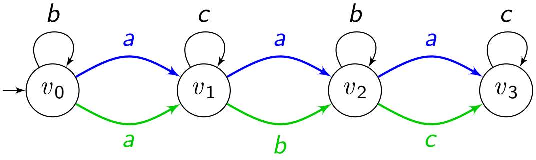


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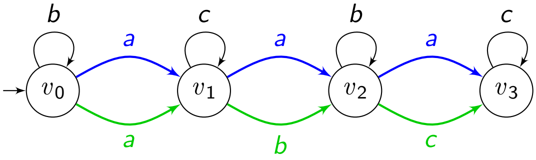


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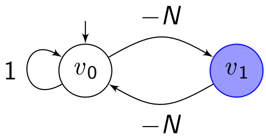
↪ **Memoryless strategies can already be too large to represent in practice!**

# Memory is not always an issue

Multi-objectives games involving payoffs often require **exponential memory**. E.g., energy-Büchi objective with $N \in \mathbb{N}$.



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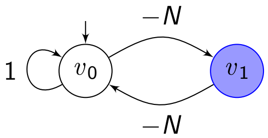


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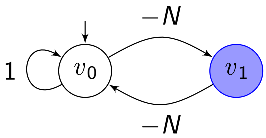


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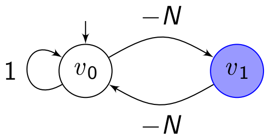


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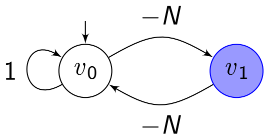


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**Hot take**

We should explore novel notions of **simplicity**, and consider *alternative representations* of strategies/controllers.

↪ **We quickly survey a few ones in the next slides.**

# Structurally-enriched Mealy machines

Idea:

- ▷ Augment Mealy machines with **data structures**: e.g., counters.[19]
- ▷ Avoid "flattening" structural information about the strategy: more succinct representations, better understandability, and closer to actual controllers.

$\Longrightarrow$ **Changes our way of thinking which strategies are complex or not.**

[19] Blahoudek et al., "Qualitative Controller Synthesis for Consumption Markov Decision Processes", 2020; Ajdarów et al., "Taming Infinity One Chunk at a Time: Concisely Represented Strategies in One-Counter MDPs", 2025.

# Decision trees

- Structured state-space (e.g., $\subset \mathbb{Z}^n$) and action-space.
- Learn a (possibly approximative) decision tree from a given **memoryless** strategy.
- More understandable and compact than huge action tables.
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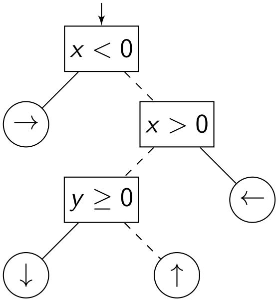
Toy example: trying to reach the center $(0, 0)$ of a 2D-grid.




**instead of**

| $x$ | $y$ | action |
|---|---|---|
| 0 | 1 | $\downarrow$ |
| 0 | 2 | $\downarrow$ |
| ... | ... | $\downarrow$ |
| $-1$ | 0 | $\rightarrow$ |
| $-1$ | 1 | $\rightarrow$ |
| ... | ... | ... |

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**Works well in practice...** [20]

**...starting from a given memoryless strategy.**

[20] Brazdil, Chatterjee, Chmelik, et al., "Counterexample Explanation by Learning Small Strategies in Markov Decision Processes", 2015; Brazdil, Chatterjee, Kretinsky, et al., "Strategy Representation by Decision Trees in Reactive Synthesis", 2018.

# Other alternatives

- **Programmatic representations.**
  - Closer to realistic code, understandable.
  - Strongly linked to the input format of the problem (e.g., PRISM code[21]), hard to generalize.
- **Models inspired by Turing machines.**
  - Powerful but hard to work with.
  - Tentative notion of decision speed.[22]
- **Neural networks.**
  - Prevalent in RL.
  - Hard to understand and verify.
  - Can be coupled with finite-state-machine abstractions.[23]

[23] Shabadi, Fijalkow, and Matricon, "Programmatic Reinforcement Learning: Navigating Gridworlds", 2025.

[23] Gelderie, "Strategy machines: representation and complexity of strategies in infinite games", 2014.

[23] Carr, Jansen, and Topcu, "Verifiable RNN-Based Policies for POMDPs Under Temporal Logic Constraints", 2020.

# Wrap-up

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**Strategy complexity $\neq$ representation complexity.**

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Complexity of strategies in controller synthesis.

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- High-level picture w.r.t. **memory** and **randomness**.

$\hookrightarrow$ **Many questions are still open!**

**Strategy complexity $\neq$ representation complexity.**

## Take-home message

We need a **proper theory of complexity**, and a **toolbox of different representations**.

$\hookrightarrow$ **Ongoing project ControlleRS.**

# Thank you! Any question?